MATHEMATICAL METHODS IN ECONOMIC RESEARCH.[1]

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Within the last few years there has been a great expansion of the use of mathematical methods in the field of economics. This expansion has taken two well-defined directions: one, that of statistical analysis of time series, the other, that of laying a mathematical foundation for the science of economics. It is the purpose of this paper to give a brief summary of the methods used, and the results obtained, by those who are carrying forward these investigations. The expansion in the field of economic statistics will be first considered.

In 1900 the Bureau of Economic Research was organized for the investigation of practical subjects in economics, statistics and politics, from a non-partisan but progressive point of view, "to the end that thoughtful men, however diverse their views of public policy, might base their discussions on objective as distinguished from subjective opinion." In 1920, the National Bureau of Economic Research was chartered. The personnel of the new organization includes that of the old and in addition many new names. The scope of the bureau has been enlarged to include "quantitative investigations into all subjects that affect public welfare." Two publications, the first of which appeared in 1921, have been issued. They contain an exhaustive report on "Income in the United States; its amount and distribution." In 1917, the Committee on Economic Research in Harvard University was appointed. Of this Committee, Professor Persons is statistician and editor. The activities of this committee have resulted in the founding of two quarterly journals, the *Review of Economic Statistics*, the first number of which appeared in January, 1919, and the *Harvard Business Review*, the initial appearance of which was in October, 1922. In addition to the two quarterlies, the Harvard Economic Service was established in 1922. This consists of a weekly letter, the principal function of which is to forecast business conditions. In addition to the above, many papers have appeared in statistical literature on price-trend curves, seasonal variation, law of growth, business cycles and related topics. Especially valuable have been some recent papers in the *Journal of the American Statistical Society*.

It may be interesting to pause for a moment, as we stand at the threshold of what apparently is to be a still greater expansion of statistical methods in the field of business, and determine what considerations have made such an expansion possible. For many years mathematicians had been spinning pennies, and testing out the laws of probability by noting the number of heads and tails that appeared; had been throwing dice and observing whether particular combinations made their appearance as often as the theory of probability indicated they should appear; had been drawing balls of many colors from urns of fixed

[1] Read at the summer meeting of the Association at Vassar College, Poughkeepsie, New York, Sept. 6, 1923.

composition in order to develop a theory of random sampling. "Urn schemata" had become a well-worn phrase. Problems arising from such considerations were, however, wholly academic, since they were artificial in character and had no connection with practical affairs. But there came a time, and to me this seems to be the reason for the tremendous activity in the field of economic statistics indicated above, when the question arose as to whether the laws of probability which had been the outgrowth of experiments artificial in character did not have a practical application. The mathematician said "yes," and the researches of the last ten years have shown that the mathematician was right in his conviction that the actions of the mass man could be forecast with the same degree of accuracy as we forecast the different combinations of heads and tails when spinning pennies.

The problem of forecasting has become one of the new problems of business. Its importance is now universally recognized and business men who gauge not only their production but their purchases of raw material by these forecasts are finding themselves in a much stronger position than those who do not. As might be expected in the treatment of such a new problem, the methods used are rather sharply differentiated. For the purposes of this paper they may be classified as: (1) empirical, (2) graphical and (3) theoretical. Upon my own responsibility and without consulting those concerned, I would name the Harvard Committee as leaders in the first method of the analysis of data, the National Bureau of Economic Research and the private service companies, such as the Babson Service, in the second method, and Mr. Fisher, author of the *Mathematical Theory of Probabilities*, in the third method. It is realized that the classification here suggested is only approximately correct, that the various methods blend one into the other, yet such a classification will serve as a basis for some remarks regarding these methods.

In the initial number of the *Review of Economic Statistics*, Professor Persons gave in detail his method of analyzing a time series and the reasons upon which his choice was based. An abstract of this method, which will be included in the forthcoming "*Handbook of Mathematical Statistics*," appeared in the June number of the *Journal of the American Statistical Society*. It may be summarized as follows:

(1) The determination of the curve of trend by means of the principle of least squares;
(2) The determination by months of the moving averages, or link relatives;
(3) The determination of the monthly medians of these values;
(4) The reduction of these medians to January as a base, in such a way as to secure a definite cycle, the error distributed by the compound interest formula.

Attention may be called to the three assumptions here involved:

(1) That the line of trend is given by a simple curve such as the linear, the parabolic or the exponential;

(2) That the median may be used to represent a series of values of the variate when the number of va ues is not large and the dispersion is considerable, and
(3) That the compound interest formula may be used to distribute errors even if they are of considerable size.

Each of these assumptions deserves consideration. The first one, that the law of growth can be closely approximated by a simple curve, seems to imply that growth itself is a simple affair. We will willingly grant this in the case of a principal at simple interest or simple discount, or in the case of a principal where the increase at any time is proportional to the principal at that time, but when we try to carry such simple laws of growth over to that of the monthly sales of automobiles or paint, we get into difficulty. The laws of growth in the demand for these two commodities cannot be so simply expressed. They are entirely different and rest upon different foundations. Paint is a necessity, the automobile is, to a great extent, a luxury. We must paint our houses in order to preserve them, and hence in order to discover the law of growth of the demand for paint we must discover the law of growth in the number of houses which must be painted. But the law of growth in the demand for automobiles is found in the accumulation of an excess of income over outgo and hence is a function of the prosperity of the country. In fact in most of the series which are worthy of study, growth is complex and it is an open question whether an attempt to express it in simple terms will in the end be satisfactory. These simple terms will not represent the law of growth with accuracy yet the method used may be as accurate as the data to which it is applied, and an attempt to use a general law of growth, or a different law for each series, may not be desirable.

Prescott [1] proceeded on the hypothesis that the law of growth passes through four distinct stages:

(1) Period of experimentation;
(2) Period of growth into the social fabric with a constantly increasing rate;
(3) Period of growth at a constantly diminishing rate, and
(4) Period of stability.

He found that a curve, due to the distinguished actuary, Benjamin Gompertz, and whose equation is

$$l_x = kg^{c^x},$$

closely approximated this cyclic growth. This is the formula which, modified by Makeham, has become a fundamental one in actuarial work. Prescott used this curve in his effort to forecast the number of automobiles that would be absorbed by the people of the United States in 1922. While the growth of any time series is usually a direct function of the population, which actuaries assume to be increasing according to a simple exponential law, yet in the periods of experimentation and stability such a law for a time series will evidently fail.

[1] Prescott, R. B., Law of Growth in Forecasting Demand. *Journal American Statistical Association*, Dec., 1922.

The second assumption, made by those who are using the empirical method in their analysis of a time series, that the median of the link relatives may be used to represent the series, raises the old question as to the value of this statistical average. In a recent note, Fisher [1] mentions the fact that its use was condemned by De Moivre, Bernoulli, and by many of the great mathematicians who have worked in the field of statistics. He also quotes Thiele and Charlier against this practice.[2] The objection is based upon the hypothesis that the errors of random sampling are so great that conclusions based upon less than a thousand values of the variate have little value. It is evident that if we are trying to find the seasonal variation of the sales of automobiles or paint, we do not have at our command statistics for anything like a thousand Januarys or Februarys. And even in those series where a complete record for many years is available, but where the unit of measurement is the dollar, we can not go back of 1918 or 1919 in any study that we may make. The question then of the choice of the method of averaging is indeed an important one. The determination of that value of the variate which has the greatest probability (the mode) is, theoretically, a comparatively simple matter. We may consider the fifteen or twenty values of the moving averages as a frequency distribution, pass through these a generalized frequency curve of the Gram-Charlier type, and determine the mode by solving the equation obtained by setting the first derivative equal to zero. The equation of this curve may be written

$$F(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{\frac{\lambda_1\omega}{1!}i + \frac{\lambda_2\omega^2}{2!}i^2 + \frac{\lambda_3\omega^3}{3!}i^3 + \cdots} e^{-xi\omega}d\omega .$$

where the semi-invariants λ_i $(i = 1, 2, 3, \cdots)$ are defined by the relation

$$e^{\frac{\lambda_1\omega}{1!} + \frac{\lambda_2\omega^2}{2!} + \frac{\lambda_3\omega^3}{3!} + \cdots}\int_{-\infty}^{+\infty}\varphi(x)dx = \int_{-\infty}^{+\infty} e^{xi\omega}\varphi(x)dx.$$

The practical difficulty of this method is that the first derivative when set equal to zero can only be solved with great labor. I have Mr. Fisher's permission to quote from a private letter: "The methods I apply, based upon the work of Laplace, Poisson, and the extensions by Gram, Thiele, and myself, do not necessarily assure us that we thus obtain the most probable value (the mode) in the strict sense of the word. Neither Laplace, Poisson, Gram or Thiele made such an assertion; they speak of the method as the most advantageous method, as the method which ought to be preferred. What we obtain is not therefore necessarily the mode but the most advantageous value of the variate." Mr. Fisher did not say how this most advantageous value of the variate is obtained, whether or not it is an approximate solution of the first derivative when a sufficient number of terms of the frequency curve has been retained to secure a good fit. Such an elaborate procedure based upon the theory of probabilities as developed by Laplace appeals to the mathematician and theorist. Yet it is quite beyond

[1] *Journal American Statistical Association*, June, 1923.
[2] A. Fisher, *Mathematical Theory of Probabilities*.

the reach of the statistician who is not a mathematician and the question arises whether it is not too theoretical, and not sufficiently responsive to world conditions.

The third assumption that the errors of random sampling can best be distributed by the compound interest formula seems reasonable. Sometimes, however, the error is quite large and appreciable changes are made by it in the coefficients of seasonal variation. It might be well to try other methods and compare results.

I do not wish to be understood, in commenting upon the assumptions made by those using empirical methods, as criticizing those methods. They have already proved their worth. Great advances have been made by their use, and we look forward to still greater accomplishments. The business executive who studies the weekly letter of the Harvard Committee has definite grounds upon which to base his decisions.

The second method used in economic research is the graphical one. A typical study, that of turpentine, as made by the statistician of a large paper company, will make this method plain. The Babson line for general business is plotted for twelve or fifteen years. The area between this curve and the curve of trend shows the volume of business above or below the normal. As an additional guide the Babson line for the ten basic commodities is drawn. A curve is now drawn showing the monthly prices of turpentine, along with others showing exports, receipts, and stock on hand. Since eighty-five per cent. of the turpentine produced is used in the paint industry, and since this industry depends upon those financial conditions which govern building, a curve showing the amount of residence building permits is also drawn. These curves are now studied in the light of world conditions at any particular time, such as shortage of receipts, unusual weather conditions, business failures, price control, transportation difficulties, painter strikes, foreign buying, labor troubles, war conditions, advances in wages, speculation, etc. This analysis of a time series which in this case is the market price of a commodity is accomplished without the use of mathematical formulæ of any kind. The statistician has before him a picture of what has taken place and attempts to interpret the future with its aid. We may urge as a criticism of this method that too great emphasis is placed upon world conditions the effect of which it is easy at the moment to exaggerate, that there is no way to care for seasonal variation and that the long time movements due to conditions other than those of the present are ignored. Yet these line pictures drawn free-hand are wonderfully illuminating and the business executive who combines their information with keen judgment and reliable current information is in a far better position to act intelligently than one who buys when the spirit moves him.

The third method used in economic research, the theoretical, attempts to solve the same problems as those to which the other two methods are applied but insists upon rigorous theoretical treatment in every case. Attention has already been called to the two ways of drawing a frequency curve, the free-hand method, and the Gram-Charlier method, also to the two ways of averaging,

the median and the mode. It may be that experience will show that the method applied must depend upon the data under consideration, not only upon the accuracy of the values of the variate but also upon their number. The mathematician is naturally prejudiced in favor of rigorous methods of analysis but he must not forget that most of the data he deals with, for instance the monthly production of yellow pine or Douglas fir, are rough, and rough methods may be preferable.

In the study of a time series one is naturally led to the comparison in terms of their standard deviations of two series in order to find out if their time variations are the same or whether like changes occur with a constant time difference, whether there is a so-called lag. In the April, 1919, number of the *Index*, it is shown that for a six months' lag, the correlation between the production of pig iron and the rates on 60–90-day paper is .75 for the years 1903–1918. When variations in one series antedate those in a second series the first becomes a barometer for the second. If we grant the hypothesis that nothing just happens, that every variation has its cause, we must admit its corollary, that barometers exist for all kinds of businesses, and their discovery therefore becomes a challenge to the statistician. The business depression of 1920–21 did not come as a stroke out of a clear sky; it came because certain fundamental economic principles had been violated and its coming we now realize should have been foreseen. It differed from preceding periods of depression only in its severity. It was preceded by over-production, over-expansion of credit, waste in manufacture, inefficiency of labor and a mounting interest rate. The researches of the last two years have shown that these conditions have always preceded the low point in the business cycle. We did not appreciate fully the value of these symptoms as the means for a financial diagnosis in 1920. We do appreciate them in 1923. These sensitive barometers would have warned us of its approach and would have enabled us to discount its effects. Every coming event must cast its shadows before and it is the function of the business statistician to recognize these shadows so that the rigors of the storm may be abated. It is the unexpected that causes the greatest damage. The business men of the country are awakening to the value of economic research, are beginning to realize that the laws of probability are holding in the business world. A new profession is being born and if this new profession rises to its highest ideals, then those seasons of business depression which in the past have been such a curse to our country will, in a large measure at least, be levelled. The trough of the wave will not be so deep; the crest not so high.

Now we may ask what part the Mathematical Association of America and the Universities which are represented in its membership are to play in the production of men for the profession of the business statistician. This question demands serious consideration. It must not be left to those who are not qualified. Through the activities of the College of Agriculture, the Ohio State University has for years been serving the farmer. Now the business men of the state are demanding that some attention be given to them. A new type of investigator

must be equipped. May I suggest that he must first of all have a broad general education. He must know economics but not to the extent that he is an idealist; he must know calculus, the theory of least squares, the theory of errors, the theory of probabilities, the theory and practice of statistical procedure. He must be a skilled computer; know when to use a slide rule and when to use a six-place logarithm table. He must have a sense for accuracy and be able to tell at a glance whether data are reasonable or not. He must know how to fit the accuracy of his methods to the accuracy of his data. He must realize that precise methods do not make inaccurate data more accurate. He must associate with business men and learn something of their psychology. He must have common sense and must draw conclusions quickly and accurately. I think I am justified in the statement that the courses in statistics as given by a majority of the universities do not meet these demands. A course in statistics based upon freshman mathematics must be to a large extent a failure. The basis for the study of statistics must be as wide and deep as the subject itself.

So far in this paper our attention has been focussed upon the statistical methods that are used in analyzing time series and upon the application of these methods to the problem of forecasting. A much more fundamental problem of economic research however is that of determining whether or not the science of economics, or possibly I had better say theoretical economics, may be placed upon a firm mathematical basis, whether a theory of economics, expressed quantitatively in terms of formulæ and equations, may be built up sufficiently simple to be verified in actual societies. We were told at our December meeting that such would have to be the case before economics could be classed as a science. In the MONTHLY (*1922*, 371–380), Professor G. C. Evans laid the foundation for such a mathematical treatment. If

$$q(u) = Au^2 + Bu + C$$

represents, as a quadratic function, the total cost of producing (that is, putting on the market) u units of a commodity in unit time and

$$y = ap + b$$

represents as a linear function the amount of goods, y, which if the price is p, will be bought in the market in unit of time, then out of these equations may be deduced conclusions as to production and prices which correspond to the varying conditions of coöperation, competition and monopoly. We see in his treatment the beginnings of a general system of economics, and look forward to the further development. I wish also to refer to other papers dealing with such questions as "Rates of Exchange" by Bray; "Elasticity of Demand and Flexibility of Prices" by Moore and "The Law of Growth in Forecasting Demand" by Prescott. The attempt to analyze the business cycle by means of Fourier series is an interesting application of that series. Such papers as I have mentioned seem to foreshadow a new day, when the economist must be first of all a mathematician. It will then be seen that the great masses of economic data which have been

accumulating for the last twenty-five years are amenable to mathematical analysis, and can be written in terms of mathematical formulæ.

I wish, in drawing this paper to a close, to give to it a personal touch. It is my privilege to be serving, in the capacity of business statistician, a corporation manufacturing paint. To the officials of such a company a forecast of the future is of the greatest importance. The president and vice-president are interested in the larger questions of development and expansion, in order that adequate preparation may be made. The general manager wants to know when to increase his invoice, and when to decrease it. He wants to know what are the readings on the paint barometer; whether the morrow will be "fair and bright," or "dark with impending storm." The factory superintendent wants to eliminate rush seasons, to do away with overtime, with all of its high over-head and inefficiency. He must satisfy all demands upon the factory, and any advance information as to what these are likely to be is of great value. The purchasing agent wants to know when to buy raw materials. The man who is spending millions of dollars of the corporation's money must know market conditions, he must buy when the market conditions are right, and he must have the most reliable information obtainable. The sales manager, who has charge of the distribution of the finished product among the various branch houses, must see to it that their invoices are properly maintained. When he ships a carload of house paint to Kansas City, the laws of probability must be depended upon to decide just how much shall go forward in five gallon containers, how much in one gallon, two quart, one quart, one pint, or one-half pint containers. If a mistake is made, it means orders shipped by express to take care of deficiencies, or it means large amounts invested in slow-moving invoices. As I make my monthly rounds of the factory, I am met always with the same question everywhere: "What is going to happen?" In order to answer this question intelligently I must keep informed as to the monthly production of lead and zinc, to the visible supply of flax seed, the production and sale of turpentine, the production and sale of construction timber, the number and value of residence building permits, employment and wages, and the buying power of the country as a whole. With this information in hand, monthly forecasts as to the sale of paint and the price trends of the raw materials may be made. The task, you will appreciate, is not an easy one.

The profession of the business statistician was born of the travail of the business depression of 1920–21. The government realizes that ignorance on the part of the business man was the underlying cause of that depression. It is now doing all it can to supply dependable information. This information must be interpreted. The Mathematical Association of America must aid in preparing and working out a curriculum which will meet the needs of the one who is to make the interpretation. The appointment of a committee who shall study this new demand upon the statistician may be desirable. A new avenue of service to his country is opening before the mathematician. He must not fail.

RAHN'S ALGEBRAIC SYMBOLS.

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1. The modifications which the algebraic symbols found in J. H. Rahn's *Teutsche Algebra*, Zürich, 1659, underwent in the English translation of that book made by Th. Brancker have never been recorded in histories of mathematics.[1] Conflicting statements occur, even among modern writers, on the question whether Rahn himself originated the symbols of 1659 or whether it was John Pell. Particularly is this true of the sign ÷ for division. An effort will be made in this article to sift the facts from conjectures and to set forth the difference between the symbols in the English and the original editions.

2. **Johann Heinrich Rahn** (1622–1676) belonged to a distinguished family in Zürich, Switzerland. His father was burgomaster and he himself held various public offices of responsibility. He says in the Dedication (Zuschrift) that in his youth he enjoyed opportunities for study both at home and abroad, but does not tell what foreign places he visited. He states in his preface (Vorbericht) that in the preceding summer he met at the watering place, Taynach, a nobleman, Leonhard Weiss, with whom he discussed algebra and to whom he promised to prepare a work in the German language which should contain the advances due to Vieta, Descartes and others, which were at that time accessible only in the Latin and French languages. He says that he took his problems partly from Vieta, Descartes, van Schooten, Diophantus and Clavius. Then comes the statement of importance to us:[2] "In the solutions and also in the Arithmetic I proceed in an entirely new way which has been used by no writer on algebra in public print, and which I first learned from a high and very scholarly person, to whom I should very gladly acknowledge indebtedness and humble respect, had he permitted."

Who was this distinguished personage? Swiss and German writers agree[3] that he was the Englishman, John Pell, who as professor at Amsterdam during 1643–1646 and at Breda in the Netherlands during 1646–1652 had distinguished himself as a lecturer on Diophantus and who during 1654–1658 served as Oliver Cromwell's political agent to the Protestant cantons of Switzerland. This conjecture is reduced to a certainty by a letter which Rahn wrote to Pell on April 17, 1675, acknowledging the receipt of a copy of the English edition of his

[1] As far as I know, the only copy of the German edition of Rahn's *Algebra* in the United States is the one of the late Professor W. W. Beman. Through the kindness of Professor L. C. Karpinski and the courtesy of the Library of the University of Michigan, that copy was placed at my disposal.

[2] Last page of "Vorbericht": "In den Solutionen, und grad auch in der Arithmetic bediene ich mich einer ganz neuen manier, die bey einichen Algebraischen Scribenten in offenem Drukk gebraucht worden, und die ich von einer hohen und sehr gelehrten Person erstmals erlehrnet hab, deren ich auch schuldiger maassen, und zwaren zur bezeugung untertähnigen respects, gar gern gedenken, so sie es hette zulassen wollen."

[3] Rudolf Wolf, "Joh. Heinrich Rahn von Zürich" in *Biographien zur Kulturgeschichte der Schweiz*, vol. 4, Zürich, 1862, pp. 55–66, 60; M.Cantor in *Allgemeine Deutsche Biographie*, Leipzig, 1888; G. Wertheim in *Bibliotheca mathematica*, 3.S., vol. 3, 1902, pp. 113–126, 116.

book. The letter begins:[1] "Most distinguished Sir. From a former student I received the algebra which in many parts is not more mine than yours, transmitted to me by your generosity, whence I infer that in the course of years your memory of me is still fresh. . . ."

From the above it appears that the peculiar arrangement of the work in the solution of equations, consisting, as we shall see, of dividing the page into three columns and registering the successive steps in the solution, is due to Pell. In this way, all the steps in the process are made evident through the aid of symbols, hardly a word of verbal explanation being necessary. The only similar arrangement known to the present writer occurs in the *Cursus mathematicus* of Pierre Hérigone, 1634 and 1644, where in a geometric proof the page is divided into two columns, one column containing the references to previous theorems used in the proof.

Rahn's book was praised by Leibniz[2] who referred to it as an "elegant algebra." Nevertheless it was not popular in Switzerland and Germany, otherwise its peculiar notation and its arrangement of solutions would have found adoption there. But neither Rudolph Wolf[3] nor the present writer has been able to find that continental writers used them.[4]

R. Wolf gives the addititional information[5] that Rahn completed in 1667 a very greatly enlarged edition under the title *Algebra speciosa*, written in Latin, in two volumes, of which the second was devoted wholly to Diophantine problems. Although urged to publish it, Rahn did not do so, perhaps because of the lukewarm reception given to his earlier work. Rahn deposited the manuscript in the public library of Zürich, "as a proof of his love" for the science. He refers to this manuscript in a letter to Mr. Haak, a member of the Royal Society in London (a letter twice mentioned in the correspondence of John Collins[6]).

3. Comparison of Symbols in the 1659 and the 1668 Editions. The symbols that are peculiar to Rahn's *Algebra* are given in the two editions as follows:

Meaning	*German edition, 1659*		*English edition, 1668*	
1. Multiplication	*	p. 7	same	p. 6
2. $a + b$ times $a - b$	$\overline{a+b \mid a-b}$	p. 14	same	p. 12
3. Division	÷	p. 8	same	p. 7
4. Cross multiplication	*×	p. 25	*X	p. 23

[1] The opening sentence is: "Vir celeberrime. A studioso quodam algebram non amplius meam sed multis nominibus tuam, mihi ex munificentia tua transmissam accepi, unde colligo, tot annorum decursu, memoriam mei adhuc apud te vigere . . . ." This letter (as G. Vacca informed G. Wertheim) is inserted in Pell's copy of the English edition of Rahn's algebra, now in the British Museum. See G. Wertheim, *loc. cit.*, p. 122.

[2] *Leibnizens Mathematische Schriften*, ed. by C. I. Gerhardt, vol. VII, p. 214.

[3] R. Wolf, *op. cit.*, p. 61 note.

[4] In Great Britain, Pell's three-column arrangement was adopted in several texts, including John Ward's *Young Mathematician's Guide*, parts of John Wallis' *Treatise of Algebra*, 1685, 1693; John Kirby's *Arithmetical Institutions*, London, 1735.

[5] R. Wolf, *op. cit.*, pp. 62, 63.

[6] S. P. Rigaud, *Correspondence of Scientific Men of the Seventeenth Century*, Oxford, 1841, vol. I, p. 154; vol. II, p. 201.

Meaning	*German edition, 1659*		*English edition, 1668*	
5. Involution	Archimedean spiral (Fig. 1)	p. 10	Ligature of omicron and sigma (Fig. 2)	p. 9
6. Evolution	Ligature of two epsilons (Fig. 1)	p. 11	same	p. 9
7. Erfüll ein quadrat / Compleat the square	E□	p. 16	C□	p. 14
8. Sixth root	$\sqrt{\text{qc.}\begin{cases}\text{aaa} = \sqrt{\text{a}}\\ \text{aa} = \sqrt{\text{c.a}}\end{cases}}$	p. 34	*cubo-cubick √ of* aaa = √a *cubo-cubick √ of* aa = √c.a	p. 32
9. Therefore	∴ (usually)	p. 53	∵ (usually)	p. 37
10. Impossible (absurd)	⊁	p. 61	ƆI	p. 48
11. Equation expressed in another way	'	p. 67	same	p. 54
12. Indeterminate, "Liberty of assuming an equation"	(*)	p. 89	same	p. 77
13. Nos. in outer column referring to steps numbered in middle column	$\dot{1}, \dot{2}, \dot{3}$, etc.	p. 3	1, 2, 3, etc.	p. 3
14. Nos. in outer column *not* referring to numbers in middle column	1, 2, 3, etc.	p. 3	$\bar{1}, \bar{2}, \bar{3}$, etc.	p. 3

4. Remarks on these Symbols. No. 4. If the lowest common multiple of *abc* and *ad* is required, Rahn writes $\frac{abc}{ad} = \frac{bc}{d}$; then $\frac{abc}{ad} * \times \frac{bc}{d}$ yields *abcd* in each of the two cross-multiplications.

No. 8. Rahn's and Brancker's modes of indicating the higher powers and roots differ in principle and represent two different procedures which had been competing for supremacy for several centuries. Rahn's √ qc. means the sixth root, $2 \times 3 = 6$, and represents the Hindu idea. Brancker's "cubo-cubick" root means the sixth root, $3 + 3 = 6$, and represents the Diophantine idea.

No. 9. In both editions occur both ∴ and ∵, but ∴ prevails in the earlier edition; ∵ prevails in the later.

No. 10. The symbols indicate that the operation is impossible or, in case of a root, that it is imaginary.

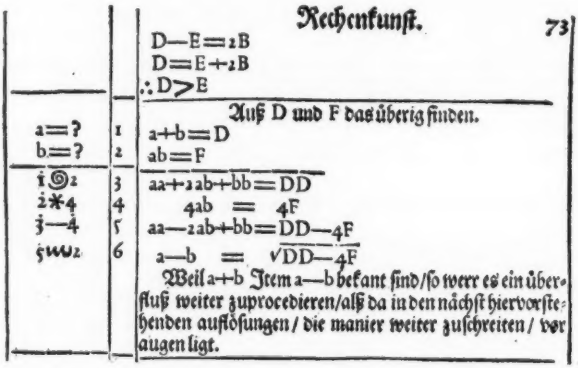
No. 11. The use of the comma is illustrated thus: The marginal column (1668, p. 54) gives "6, 1" which means that the sixth equation "Z = A" and the first equation "A = 6" yield Z = 6.

No. 12. For example, if in a right triangle *h*, *b*, *c*, we know only $b - c$, then one of the three sides, say *c*, is indeterminate.

5. Preparation of the English Edition. Thomas Brancker, the translator, was then a young man twenty-five years of age, a graduate of Exeter College, Oxford, who had studied chemistry under Robert Boyle and had written a booklet on the doctrine of the sphere and globe. In his preface of the translation Brancker says:

"The copy [of Rahn] which I have, was given me anno 1662. By a good Friend, [M.F.T.] who then told me he much desired to read it in some Language that he understood; I then promised him to *English* it. . . . When it was finished, I desired to see it Printed, and got it Licensed May 18. 1665. . . . And so without any alteration either in the Precepts or Examples, save

only the correction of many Mistakes: It was sent to the Press, . . . A little after, I heard that there was at that time in London, a Person [D.I.P.] of Note very worthy to be made acquainted with my design, before I made any farther progress in the Impression. Being admitted to speak with him, I found him not only able to direct me, but also very willing so to do, so far as his leisure would permit. He gave me divers cautions concerning the Work. He shewed me the way of making the Table of Incomposits, of examining it, and of continuing as far as I would. He encouraged me to extend it to 100 thousand: Telling me that by that time that I had Calculated and Printed that Table, he hoped to be at leisure to review some of Monsieur Rhonius his Problemes, and to work them anew; and that he would send them to me, with leave to publish them or to keep them by me. I had finished and Printed that Table, as also Twelve sheets of the Book itself, before he sent me his Alterations. They begin with Probl. 24. pag. 100. All from thence to the end is his Work: As also pag. 79. 80. 81. 82. which he sent last of all: So that instead of the first 124 pages of Rhonius, this hath just twice as many."



Rechenkunſt. 73

D—E=1B
D=E+1B
∴ D>E

Auß D und F das übrig finden.

a=?	1	a+b=D
b=?	2	ab=F
i◎2	3	aa+2ab+bb=DD
i✳4	4	4ab = 4F
j—4	5	aa—2ab+bb=DD—4F
ȷwu2	6	a—b = √DD—4F

Weil a+b Item a—b bekant ſind/ſo weere es ein überfluß weiter zuprocedieren/alß da in den nächſt hiervorſtehenden auflöſungen/ die manier weiter zuſchreiten/ vor augen ligt.

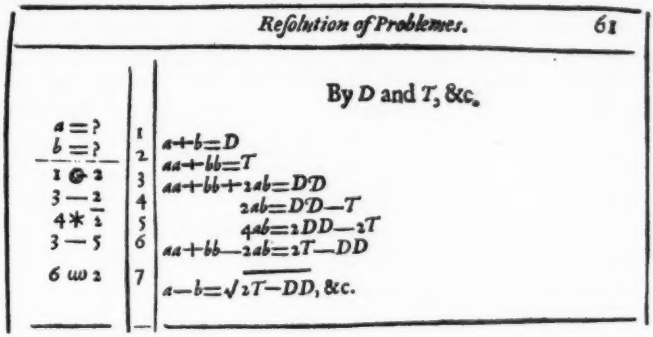
FIG. 1. From Rahn's *Teutsche Algebra*, 1659. This shows: (1) The first use of ÷ in print, as a sign of division, which occurs in various places and also on page 73, but below the part of that page reproduced here, (2) the Archimedean spiral for involution, (3) the double epsilon for evolution, (4) the use of capital letters *B*, *D*, *E*, for given numbers, and small letters *a*, *b*, for unknown numbers, (5) the ✳ for multiplication, (6) the first use of ∴ for "therefore," (7) the three-column arrangement, of which the outer column contains the directions, the middle the numbers of the lines, the inner the results of the operations. Thus, in line 3, we have "line 1, raised to the second power, gives $aa + 2ab + bb = DD$."

We have copied so much of Brancker's preface to his edition of Rahn, because it explains in a simple way how he came to undertake the translation, how and at what stage of the publication he came in contact with that "Person of Note very worthy" who was no other than John Pell.

Rahn's name does not appear on the title page of the English edition.[1] The Latinized form of his name, Rhonius, is given in Brancker's preface. Nor does John Pell's name appear in the book, except by the letters "D.P." [Doctor Pell] on the title-page, and the letters "D.I.P." in the margin of the Preface.

[1] The full title of the English edition is as follows: "An Introduction to Algebra, translated out of the High-Dutch into English, by Thomas Brancker. M.A. Much Altered and Augmented by D. P. Also A Table of Odd Numbers less than One Hundred Thousand, shewing Those that are Incomposit, And Resolving the rest into their Factors or Coefficients, etc. Supputated by the same Tho. Brancker. London, printed by W.G. for Moses Pitt at the White-Hart in Little Britain. 1668."

6. Growth of Legend. A series of misstatements came to be printed, in course of time in German as well as in English publications, which misstated the manner in which the English edition was prepared for the press, gave Pell the credit for all the meritorious qualities of the book and in some cases even charged Rahn with plagiarism. All these misstatements happened through no fault of Pell or of Brancker.



Reſolution of Problemes. 61

By *D* and *T*, &c.

a = ?	1	
b = ?	2	*a*+*b*=*D*
1 ⊙ 2	3	*aa*+*bb*+2*ab*=*DD*
3 — 2	4	2*ab*=*DD*—*T*
4 ✱ 2	5	4*ab*=2*DD*—2*T*
3 — 5	6	*aa*+*bb*—2*ab*=2*T*—*DD*
6 ω 2	7	$a-b=\sqrt{2T-DD}$, &c.

FIG. 2. From *Brancker's translation of Rahn, 1668.* The same arrangement of the solution as in 1659, but the omicron-sigma takes the place of the Archimedean spiral; the ordinal numbers in the outer column are not dotted, while the number in that column which does not refer to steps in the middle column carries a bar, $\overline{2}$. Step 5 means, "line 4, multiplied by 2, gives $4ab = 2DD - 2T$."

In the first place, it has been assumed that it was John Pell who invited Brancker to make the translation;[1] Brancker himself states that it was a friend "M.F.T." and that he got in touch with Pell after he had secured the license for the printing of the translation.

In the two-page review of the English edition in the *Philosophical Transactions*[2] no undue credit is given to John Pell. In fact he is referred to only in general terms and not by name; no specific mention is made of the part of the English edition written by him. However, the review contained the first of a series of blunders in the statement relating to prime numbers, "No Book but this, extends it to above Ten Thousands"; the original German edition of Rahn contained a table as far as 24,000.

When the English edition was in press, John Collins[3] wrote the Scotsman, James Gregory: "One John Henry Rhonius published an Algebra in high Dutch. He was Dr. John Pell's scholar; the book is translated into English, refined by the Doctor, and almost out of the press." There is nothing in the statements of Rahn or Brancker which justifies the use of the word "scholar" in the ordinary sense. In 1665 Collins had seen[4] Rahn's algebra of 1659 and probably based

[1] G. Wertheim, *loc. cit.*, p. 121, points out that Moritz Cantor, in *Vorlesungen über Geschichte der Mathematik*, vol. 2, 2d ed., Leipzig, 1913, p. 777, and also H. Konen in *Geschichte der Gleichung* $t^2 - Du^2 = 1$, Leipzig, 1901, p. 33, are among those who make this unwarranted assumption.

[2] *Philosophical Transactions*, vol. 3, London, 1668, pp. 688–690.

[3] S. P. Rigaud, *op. cit.*, vol. II, p. 177.

[4] S. P. Rigaud, *op. cit.*, vol. I, p. 135.

the use of the term "scholar" on Rahn's acknowledgment of indebtedness to Pell, which we quoted above. At least eight times [1] did Collins use the term "scholar" in his correspondence. He also refers to the English edition of Rahn as the "Dr.'s book," *i.e.*, Pell's book.[2] Likewise, Isaac Barrow, on May 14, 1668, acknowledges the receipt from Collins of "the remainder of D.P. his book." [3]

Collins refers to the English edition in his correspondence with John Wallis, professor of mathematics at Oxford. In the Latin edition [4] of his *Algebra* (1693), Wallis speaks of Rahn as a "discipulus" of Pell. In the English [5] (1685) as well as the Latin editions of that treatise, Wallis not only proceeds on the general assumption that the entire book was really John Pell's, but advances to the specific claim that all the new symbols occurring in Rahn's book are Pell's. Wallis says: [6] "The notes or Marks he [Pell] useth to this purpose are beside those of Oughtred" [7] $+$, $-$, $\times$, $::$, $\surd$, and those of Harriot $>$, $<$, "some peculiars of his own, as $\div$ for Division, and $\mathfrak{D}$, $\mathfrak{cc}$ for Involution and Evolution (as he calls them)." In another place Wallis says: [8] "Dr. John Pell, (as we shall see hereafter) adds further a Note of Division $\div$ (Divided by), which others express Fraction-wise . . . ." And again later, [9] "And using Dr. Pell's Note of Division $\div$ to avoid Double Lines . . . ." Not a shred of evidence does Wallis advance in support of his statements. These symbols were used by Rahn in print in 1659; with some modifications they were used again in 1668 by Brancker and Pell. No evidence has been advanced by any one that Pell used these symbols before 1668 in print, or in manuscript except when he prepared his alterations and additions to pages 79–82, and to what follows page 100 in Brancker's translation. The last part of Rahn's original book was omitted from the English edition.

Wallis's high rank as a mathematician and the undoubted value of many of his historical expositions have thrown some writers off their guard on certain matters of priority between English and continental mathematicians. Thus it happens that Charles Hutton [10] unreservedly ascribes the sign $\div$ to Pell. According to R. Wolf, similar statements are made by Kästner and Doppelmayr in Germany. Miss A. M. Clerke [11] says that Pell "is remembered chiefly by his invention of the sign $\div$ for division and a mode of marginally registering the successive steps in the reduction of equations."

A recent writer, Giovanni Vacca,[12] in an otherwise exceedingly meritorious

[1] S. P. Rigaud, *op. cit.*, vol. I, pp. 135, 154, 198, 243; vol. II, pp. 177, 195, 201, 472.
[2] S. P. Rigaud, *op. cit.*, vol. I, p. 135; vol. II, pp. 468, 470.
[3] S. P. Rigaud, *op. cit.*, vol. II, p. 65.
[4] J. Wallis, *Opera*, Oxford, vol. 2, 1693, p. 234.
[5] J. Wallis, *Treatise of Algebra*, London, 1685, p. 214.
[6] J. Wallis, *op. cit.*, p. 219.
[7] Rahn and Brancker did not use $\times$ for multiplication as claimed by Wallis; in the 1659 and 1668 editions of Rahn, the sign of multiplication is the six pointed star *.
[8] J. Wallis, *Treatise of Algebra*, 1685, p. 127.
[9] J. Wallis, *op. cit.*, p. 227.
[10] Charles Hutton, *Mathematical Dictionary*, London, 1796, Art. "John Pell."
[11] Miss A. M. Clerke, "Pell, John" in the *Dictionary of National Biography*.
[12] Giovanni Vacca in *Rivista di matematica*, vol. 6, 1899, p. 122.

article, not only gives Pell credit for all the new symbols in Rahn's book, but designates both the English and German editions as being Pell's book. Vacca says, "I intend in this note to give account of a symbolism of the same kind, but more complicated and precise, due to John Pell (1610–1685), and used by him systematically in his work, *Introductio in Algebram*, Londini a. 1668 (a first edition appeared in the German language, Tigurii, 1659). It has not been possible for me to consult the original work of Pell. I limit myself therefore to giving an idea of his method as it is found in the exposition given by Wallis in Vol. 2. p. 233–246 of his *Works* (London a. 1693)." Still more recently, J. Tropfke in one place [1] mentions $\div$ as used by John Pell and William Jones; in another place [2] he ascribes $\div$ to Pell, but mentions Rahn as having used it in 1659.

7. Conclusion. Nowhere have I been able to find any reason for assigning to Pell the new signs found in Rahn's algebra of 1659. There is no evidence that Pell used them before he prepared his additions to the English edition of Rahn's book. That the new signs in Rahn's algebra of 1659 are due to Rahn and not to Pell would seem to follow also from the fact that Pell used Brancker's modifications of the 1659 symbols. The symbol for involution, as used by Rahn, cleverly expressed the idea by its very shape—an Archimedean spiral. It must have been adopted because of its suggestiveness and on that account would not be likely to have been discarded by its inventor, in favor of the omicron-sigma which by its shape does not express the idea of "involviren."

The growth of myth relating to Pell and to Rahn's text is perhaps the less surprising when one recalls that erroneous statements have been current on the actual content of the two editions. As previously stated, the table of composite and prime numbers in the 1659 edition of Rahn's Algebra was said not to go beyond 10,000, when as a matter of fact the table extends to 24,000. Again, it has been said that the English edition (1668) contained a solution of the famous indeterminate equation $x^2 - ay^2 = 1$, when no such solution is actually given. Moreover, this famous equation has been called "Pell's Equation," even though Pell published nothing on it and, as far as known, did not occupy himself with it.

THE THREE-BAR CURVE.

By F. V. MORLEY, New College, Oxford, England.

1. The main attraction of the three-bar curve lies in the simplicity of its mechanical generation. Historically, the "sweet simplicity," as James Watt says, especially took his fancy in the so-called "parallel-motion" by which he drew the curve; and the "delicate smoothness" of that three-bar linkwork gave him more pleasure than any of his other inventions.[3] But though we can construct the curve mechanically with an ease exceeded only by that of the circle—

[1] J. Tropfke, *Geschichte der Elementar-Mathematik*, vol. 2, 2d ed., 1921, p. 20.
[2] J. Tropfke, *op. cit.*, vol. 2, 2d ed., 1921, p. 25.
[3] See Muirhead, *Life of James Watt*, London, 2d ed. (1859), p. 288.

I find it as easy to draw accurately as a conic—no analytical treatment of corresponding simplicity has been attained. The question is really one of providing a kinematical treatment for what is essentially a matter of elementary kinematics.

The simplicity of generation and the difficulty of analysis have attracted many writers,[1] and some very pretty results have cropped up as the outcome of their researches. In 1875, during the first stage in the history of the curve, when there was a remarkable interest in linkages, Roberts made the discovery which Cayley elaborated of the "triple-generation" of the three-bar sextic. The subject slept for some time, but during the last few years Colonel Hippisley announced a "pairing" on the curve which Dr. Bennett shaped into a keystone property—the simple transformation, by isogonal conjugates, of the curve into itself. It seems worth while to give a direct proof of this last property, based on an analysis which gives a connected treatment of the curve.[2] We have first to illustrate the type of argument; then recapitulate the "triple-generation"; and proceed to the transformation as the result of a simple algebraical theorem.

2. We shall adopt the notation and methods of inversive geometry, using that term in Professor Morley's sense as the geometry which illustrates the theory of functions of a complex variable in much the same way that projective geometry illustrates the analytic theory of forms. Here the reference scheme is a *base line*, on which are spread the real numbers ρ, and a unit or *base circle* with centre at 0. On the base circle are spread a series of special complex numbers, called *turns*, and represented by t; they are of the form

$$t = 1^p,$$

where p is any real number. This scale of turns upon the base circle has certain special properties—the product of two turns is a turn; the sum of two turns is not in general a turn; but any root of a turn is a turn. If we connect t, thought of as a point of the base circle, with the $\angle\ 10t$ (called θ), we have

$$t = \cos\theta + i\sin\theta = e^{i\theta}.$$

The conjugate complex number is

$$1/t = \cos\theta - i\sin\theta = e^{-i\theta}.$$

We express this shortly by saying the conjugate of t is $1/t$. Both t and its conjugate, $1/t$, are numbers in the scale of turns; that is, are points of the base circle.

The product of any real number ρ and any turn t is a complex number which we call

$$x = \rho t \qquad (2.1)$$

[1] Without giving a complete bibliography, one should mention Samuel Roberts, *Proc. London Math. Soc.*, vol. 2 (1869), p. 125, and *ibid.*, vol. 7 (1876), p. 17; Cayley, *ibid.*, vol. 3 (1870), p. 100, and vol. 7 (1876), p. 136; Sylvester, in several discursive papers, but one particularly to be referred to in *Nature*, vol. 12 (1875), p. 214; Clifford, *Kinematic* (1878), p. 149; Darboux, *Bulletin des Sciences Mathématiques*, ser. 2, vol. iii (1879), p. 109; and of more recent writers, Colonel Hippisley, *Proc. London Math. Soc.*, ser. 2, vol. 18 (1917), p. 136, and *ibid.*, vol. 21 (1922), p. 410; Dr. G. T. Bennett, *ibid.*, ser. 2, vol. 20 (1920), p. 73.

[2] For a condensed exposition of this analysis, see *Proc. London Math. Soc.*, ser. 2, vol. 21 (1921), p. 140.

and we identify this number x with a point of the plane. Drawing the vector $0x$, we say that ρ is the length, and t the direction factor. Thus without losing its representation of a point on the base circle, t may also be thought of as the rotation by which the line $0x$ is obtained from the base line. Equation (2.1), the naming of a point by the vector $0x$, may be looked upon as a parametric equation either of a line or of a circle, according as t or ρ is made a constant. It may readily be generalized. We follow the convention of expressing any fixed point of the plane by a constant complex number denoted by a small letter. Thus

$$x - c = \rho t \tag{2.2}$$

expresses that the vector cx has length ρ, direction t. If t is fixed, (2.2) is the equation of a line in terms of the parameter ρ. If ρ is fixed, it is the equation, with parameter t, of a circle with centre c and radius ρ.

The process of forming the conjugate of a complex number x is thought of as the operation of reflexion in the base line—an operation sending each point x into a new point which we write with a superposed bar, $\bar{x} = \rho/t$. The reflexion affects all points of the plane, the base line being invariant under the operation; it sends the base circle into itself, replacing t by $1/t$; and it sends a vector $x - c = \rho t$ into $\bar{x} - \bar{c} = \rho/t$.

3. The above outline does not take us very far, but we may indicate the form taken in this analysis by the transformation by isogonal conjugates. This transformation has three fixed points, say a_1, a_2, a_3, taken on the base circle. It will be useful to consider the points as roots of the cubic equation

$$a^3 - \sigma_1 a^2 + \sigma_2 a - \sigma_3 = 0,$$

where

$$\begin{aligned} \sigma_1 &= a_1 + a_2 + a_3, \\ \sigma_2 &= a_2 a_3 + a_3 a_1 + a_1 a_2, \\ \sigma_3 &= a_1 a_2 a_3. \end{aligned}$$

The usual defining property of a pair of points x, y under the isogonal transformation is

$$\angle\, a_2 a_1 x = \angle\, y a_1 a_3 \tag{3.1}$$

and so for the other two sets of angles (Fig. 1). We are therefore concerned with the relative directions of the vectors $a_2 a_1$ and $x a_1$. If $a_2 - a_1 = \rho_1 t_1$ and $x - a_1 = \rho_2 t_2$, the relative direction is given by

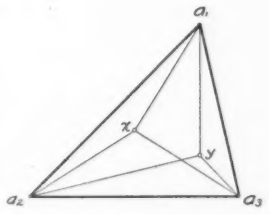


FIG. 1. The Transformation by Isogonal Conjugates.

$$\frac{x - a_1}{a_2 - a_1} = \frac{\rho_2 t_2}{\rho_1 t_1} = \mu\tau$$

where $\mu = \rho_2/\rho_1$ is real and $\tau = t_2/t_1$ is a turn. Similarly $(a_3 - a_1)/(y - a_1) = \mu'\tau'$ where, by (3.1), $\tau' = \tau$. Hence we have the expression

$$\frac{(x - a_1)(y - a_1)}{(a_2 - a_1)(a_3 - a_1)} \tag{3.2}$$

equal to a real number, μ/μ', and consequently equal to its conjugate. The congugate of (3.2), bearing in mind that $\bar{a} = 1/a$, is

$$\frac{a_1^2 a_2 a_3(\bar{x} - 1/a_1)(\bar{y} - 1/a_1)}{(a_2 - a_1)(a_3 - a_1)}.$$

Hence

$$a_1(x - a_1)(y - a_1) = \sigma_3(1 - \bar{x}a_1)(1 - \bar{y}a_1). \tag{3.3}$$

Similarly-formed equations are to be true for a_2, a_3, as well as for a_1. We may include all three equations under the form

$$a(x - a)(y - a) - \sigma_3(1 - \bar{x}a)(1 - \bar{y}a) \doteq 0, \tag{3.4}$$

where, as a takes in succession the specific values a_1, a_2, a_3, we get the equations of type (3.3). Equation (3.4), as a cubic in a, must be identically

$$a^3 - \sigma_1 a^2 + \sigma_2 a - \sigma_3 = 0.$$

Equating coefficients, we have

$$x + y + \sigma_3 \bar{x}\bar{y} = \sigma_1 \tag{3.5}$$

as the equation expressing the transformation by isogonal conjugates.

Equation (3.5) gives for every x a definite partner y, with these exceptions:

(*1*) when x is one of the a's, in which case y may be any point on the opposite side of the base triangle a_1, a_2, a_3. We avoid ambiguity by assigning points on the side, say a_2a_3, according to the directions of lines on a_1—that is, according as x approaches a_1.

(*2*) when $x \to \infty$, in which case the partner may be found by considering the path which x takes to ∞. That is, if $x/\bar{x} = \tau$ as $x \to \infty$, then $y = \sigma_3/\tau$, a point on the base circle.

The fixed points of the transformation are the centres of the four circles touching the sides of the triangle. The pair x, y are foci of a conic inscribed in the triangle; consequently the transformation (3.5) is sometimes called the focal transformation, or simply "focal pairing."

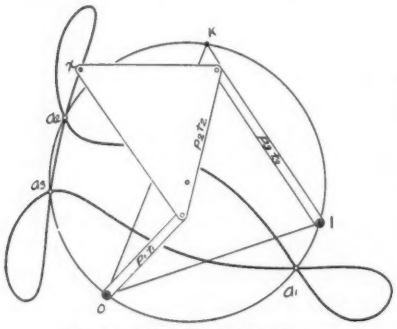


FIG. 2. The Three-Bar Curve.

4. The three-bar curve is generated by a linkwork of three jointed rods (Fig. 2), the two legs having each a pivot at fixed points of the plane. We may choose these points to be 0 and 1. The linkwork has one degree of freedom; any point on either leg traces a circle; but any point on the middle or traversing bar traces a three-bar sextic curve. Watt chose to study the motion of the midpoint of the traversing bar—a special three-bar curve. It is more general and no more difficult to consider the tracing point, x, as attached to the traversing bar by a rigid triangular plate.

If we call the lengths of the bars in order of size ρ_1, ρ_2, ρ_3 and their directions t_1, t_2, t_3, we have immediately two fundamental vector identities,

$$\rho_1 t_1 + \rho_2 t_2 + \rho_3 t_3 = 1 \tag{4.1}$$

and its conjugate equation

$$\rho_1/t_1 + \rho_2/t_2 + \rho_3/t_3 = 1. \tag{4.2}$$

In these equations the ρ's are constant and the t's vary with the motion of the linkwork. The tracing point x may be named from the base point 0 as

$$x_0 = \rho_1 t_1 + k\rho_2 t_2, \tag{4.3}$$

where k is a complex constant, giving the "shape" of the triangular plate on the traversing bar $\rho_2 t_2$. But, named from the point 1, x will be

$$\begin{aligned} x_1 &= x_0 - 1 \\ &= \rho_1 t_1 + k\rho_2 t_2 - \rho_1 t_1 - \rho_2 t_2 - \rho_3 t_3 \\ &= \rho_2 t_2 (k-1) - \rho_3 t_3. \end{aligned} \tag{4.4}$$

It is thus suggested that if we are to name x symmetrically, from base point anywhere, we should use an expression

$$x = c_1\rho_1 t_1 + c_2\rho_2 t_2 + c_3\rho_3 t_3, \tag{4.5}$$

where the three c's are complex constants.

In that case, consider each c as a fixed point of the plane. Since the identity (4.1) holds, we may write (4.5) as

$$x - c_3 = (c_1 - c_3)\rho_1 t_1 + (c_2 - c_3)\rho_2 t_2.$$

This gives the naming of x with respect to c_3. It must therefore be identical with (4.3) when $c_3 = 0$ and $c_1 = 1$. That is, $c_1 - c_3 = 1$ and $c_2 - c_3 = k$. There is then a third point k, which plays a part equivalent to that played by 0 or 1. This point is the vertex of a triangle on 01 similar to the triangle attaching x to the traversing bar. Hence we have the triple-generation theorem, that the curve traced by a three-bar linkwork attached to two fixed points, c_3, c_1, may also be traced by three-bar linkworks attached to two other pairs of fixed points, c_1, c_2, and c_2, c_3.

When Roberts communicated this theorem to Cayley, the latter put it into the form illustrated by Figures 3 and 4. That is, we take any triangle $C_1C_2C_3$ and through any point x within it draw lines parallel to the sides. Let the triangles outlined in Fig. 3 be supposed rigid plates, jointed at x, and let the other lines of the figure represent bars completing three jointed parallelograms. Then, however the system is moved about in its plane, as shown in Fig. 4, the triangle $c_1c_2c_3$ will be always of the same shape; and further, starting as in Fig. 4 from any given position of the three triangular plates, the assemblage may be moved without altering $c_1c_2c_3$ in magnitude. Unless $c_1c_2c_3$ are in their maximum position $C_1C_2C_3$, x is movable and will describe a three-bar curve under the simultaneous guidance of the three three-bar linkworks.

5. Let us call the three fixed points $c_1c_2c_3$ *poles* [1] of the three-bar curve, and consider them as on the base circle. They are then constant turns. We have

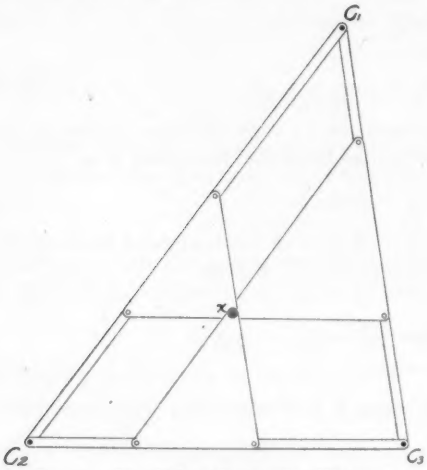


FIG. 3. The Triple Generation Theorem. Maximum Position.

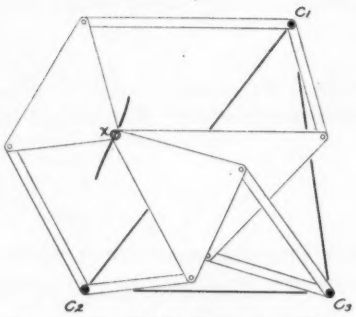


FIG. 4. The Triple Generation Theorem. In Motion.

defined the curve itself by the parametric equation (4.5), which we may write for brevity

$$x = \Sigma\rho_1 c_1 t_1, \tag{5.1}$$

the summation understood to be for three terms. This carries with it the conjugate equation

$$\bar{x} = \Sigma\rho_1/c_1 t_1. \tag{5.2}$$

Now, according to Colonel Hippisley's suggestion, let us define a new point, y, by the equations

$$y = \Sigma\rho_1 c_1/t_1, \tag{5.3}$$

$$\bar{y} = \Sigma\rho_1 t_1/c_1. \tag{5.4}$$

The point y is also on the curve.

Out of the four equations (5.1)–(5.4) we expect some simple relation between x and y. But before proceeding further we may note, from the fundamental identities

$$\begin{aligned} 1 - \rho_1 t_1 &= \rho_2 t_2 + \rho_3 t_3, \\ 1 - \rho_1/t_1 &= \rho_2/t_2 + \rho_3/t_3, \end{aligned} \tag{4.2}$$

that

$$\begin{aligned} \rho_2\rho_3(t_2/t_3 + t_3/t_2) &= 1 + \rho_1^2 - \rho_2^2 - \rho_3^2 - \rho_1(t_1 + 1/t_1) \\ &= B_1 - \rho_1(t_1 + 1/t_1), \end{aligned}$$

[1] They have also been called *foci* of the sextic; the change of terminology is not very happy, but is necessary in considering the curve from an inversive rather than from a projective standpoint. The argument is set forth in the *Proc. London Math. Soc.*, ser. 2, vol. 21 (1921), pp. 153, 154.

where B_1 is real and constant. We have then, upon forming the product xy,

$$\begin{aligned} xy &= c_1^2\rho_1^2 + c_2^2\rho_2^2 + c_3^2\rho_3^2 + c_1c_2B_3 + c_2c_3B_1 + c_3c_1B_2 - \Sigma c_2c_3\rho_1(t_1 + 1/t_1) \\ &= A - c_1c_2c_3(\bar{x} + \bar{y}), \end{aligned} \tag{5.5}$$

where A is constant.

From the definitions of x and y we therefore obtain directly the two equations

$$\bar{x} + \bar{y} + xy/s_3 = \overline{C} \tag{5.6}$$

and

$$x + y + \bar{x}\bar{y}s_3 = C, \tag{5.7}$$

where s_3 is written for the product $c_1c_2c_3$ and $\overline{C} = A/s_3$ so that C is still a constant.

The equations (5.6) and (5.7) give a one-to-one correspondence, mutual or involutory, between x and y. Thus the whole curve has a transformation into itself in which the points x, y figure as correspondent points; and the form of (5.7) instantly suggests the transformation by isogonal conjugates given in equation (3.5). In fact, we may identify (5.7) as the transformation by isogonal conjugates with respect to the three points a_1, a_2, a_3 whose symmetric functions are σ_1, σ_2, σ_3, if we put

$$\sigma_3 = s_3 \qquad \text{and} \qquad \sigma_1 = C. \tag{5.8}$$

If this is done, we arrive by a short route at the keystone property of the three-bar curve, the transformation noted first by Dr. Bennett. The points a_1, a_2, a_3 are not the same as the points c_1, c_2, c_3, but they too are on the base circle; and are, as a matter of fact, the double points of the three-bar curve. The relation (5.8), that the sum of the central vectorial angles is the same (mod 2π) for the set of double-points as for the set of poles, was given by Cayley.

THE DYNAMICS OF MONOPOLY.

By G. C. EVANS, The Rice Institute, Houston, Texas.

1. Introduction. It is related to a familiar symptom of "prosperity" that the demand for a commodity is greater when the price is increasing than when it is decreasing. An exaggerated instance of this law of demand, perhaps an undue generalization from the last crisis, is what the wholesale lumber dealers tell us is characteristic of their sales, namely, that when prices are going up the demand is insatiable, but when prices go down it is nil until the price movement stops. It is this phenomenon, far outside the scope of traditional economic theory, which we wish to discuss.

In an earlier paper (*1922*, 371–380), the author investigated the situation in which the cost of producing an amount u of a given commodity per unit time was a quadratic function of the amount produced:

$$q(u) = Au^2 + Bu + C,$$

and the demand for the commodity—the amount which the market would absorb in unit time if the price were p—was a linear function of the price:

$$y = ap + b.$$

The simplest approximation for the situation now to be discussed is to write for the demand function the following linear differential expression:

$$y = ap + b + h\frac{dp}{dt}. \tag{1}$$

The practical case is probably to take $h > 0$, but it is interesting, at least theoretically, to leave the sign of that constant arbitrary, as we shall see. As for the other constants, we assume, as in the earlier paper,

$$a < 0, \qquad b > 0, \qquad A > 0, \qquad B > 0, \qquad C > 0.$$

As before, also, we assume that $u = y$. We limit ourselves to the case of a single producer.

We must now introduce some kind of extremal condition to determine the unknown function $p(t)$. On account of the presence of the quantity $p' = dp/dt$ we cannot simply say, as we did without this quantity, that the profit

$$\pi = py - q(u) = pu - q(u)$$

is to be a maximum, for the problem has now become one of the type indicated in the closing section of the article cited, where the quantities involve a whole interval of time. A simple assumption, descriptive of monopoly, is *to make a maximum the total profit over an interval of time. Let us further assume that at the time* $t_1 = 0$ *the price is* p_1, *and at the time* t_2 *the price is* p_2; *and in the first instance restrict our investigation to functions* $p(t)$ *which are continuous with their first derivatives in the intervening stretch of time.* We can imagine that at the initial time the cost function has been changed from some previous formula to the one given above, and the producer desires, on the basis of the new cost function, to arrive at a new equilibrium price at the time t_2, by continuous change, in the most profitable manner. This is a problem of *economic dynamics.*[1]

2. Maximum of an Integral. Our task is to find the function $p(t)$ which makes a maximum the integral

$$\begin{aligned} \Pi &= \int_0^{t_2} \pi(p, p', t)dt \\ &= \int_0^{t_2} \{p(ap + b + hp') - A(ap + b + hp')^2 - B(ap + b + hp') - C\}dt \end{aligned} \tag{2}$$

with $p(0) = p_1$ and $p(t_2) = p_2$. An editor of the MONTHLY—Professor Bennett—has said that one should be obliged to present a certificate of character before

[1] See for instance the classification in Amoroso, *Lezioni di Economia Matematica*, Bologna (1921), ch. 5.

being initiated into the mysteries of the Calculus of Variations, to which study our present investigation belongs, since its fascination is so great that neophytes seek to introduce it into problems which would otherwise be perfectly simple. We shall perform, however, the reverse process and discuss the possible maxima of our integral, Π, in terms of the elementary properties of a function of a single variable.[1]

Let $p(t)$ be the desired function, if there is any which satisfies the conditions, and let $f(t) = p(t) + \psi(t)$ be any other function continuous with continuous derivative and such that $f(0) = p(0), f(t_2) = p(t_2)$. Then $\psi(t)$ will be continuous with its derivative, and $\psi(0) = \psi(t_2) = 0$. Write

$$\xi(t) = p(t) + x\psi(t),$$
$$\frac{d\xi}{dt} = \xi'(t) = p'(t) + x\psi'(t),$$

x being an arbitrary parameter, $0 \leq x \leq 1$. Consider the integral

$$\Pi(x) = \int_0^{t_2} \pi(\xi, \xi', t)dt,$$

which for a given arbitrary $\psi(t)$ or $f(t)$ is a function of the single variable x.

On account of the explicit form of $\pi(\xi, \xi', t)$ it is seen at once, by carrying out the differentiation, that, given $\psi(t)$, the quantity $d\Pi/dx$ exists for every value of x. In fact

$$\begin{aligned} \frac{d\Pi}{dx} &= \int_0^{t_2} \left\{ \frac{\partial \pi}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial \pi}{\partial \xi'}\frac{\partial \xi'}{\partial x} \right\} dt \\ &= \int_0^{t_2} \left\{ \frac{\partial \pi}{\partial \xi}\psi(t) + \frac{\partial \pi}{\partial \xi'}\psi'(t) \right\} dt. \end{aligned} \tag{3}$$

Hence it is necessary, in order that $x = 0$ shall correspond to the maximum of Π (*i.e.*, that $p(t)$ shall be the desired function), that

$$\frac{d\Pi}{dx} = 0 \qquad \text{when} \quad x = 0. \tag{a}$$

It is sufficient for a maximum if, no matter what $\psi(t)$ has been given, subject to the enunciated conditions, we have $\Pi(1) < \Pi(0)$. But by the law of the mean:

$$\Pi(1) - \Pi(0) = \frac{d\Pi(x)}{dx}$$

for some x, $0 < x < 1$. Hence it is sufficient for a maximum, if given $\psi(t)$ arbi-

[1] It is perhaps unnecessary to remind the reader that a readable introduction to the Calculus of Variations may be found in Goursat, *Cours d'Analyse*, vol. III, Paris (1915). In our problem the t does not happen to occur explicitly in the function under the integral sign.

trarily in its domain we have

$$\frac{d\Pi}{dx} < 0 \qquad \text{for all } x, \quad 0 < x < 1. \tag{b}$$

It happens that the conditions (*a*), (*b*) yield the solution of our problem.

In the expression for $d\Pi/dx$, equation (3), we can now perform on the term involving $\psi'(t)$ an integration by parts and write

$$\frac{d\Pi}{dx} = \left[\frac{\partial \pi}{\partial \xi'}\psi(t)\right]_0^{t_2} + \int_0^{t_2}\left\{\frac{\partial \pi}{\partial \xi} - \frac{d}{dt}\frac{\partial \pi}{\partial \xi'}\right\}\psi(t)dt, \tag{3'}$$

in which the first term of the second member vanishes, since $\psi(0) = \psi(t_2) = 0$. By a short calculation

$$\frac{\partial \pi}{\partial \xi} = 2a\xi(1 - Aa) + (b - 2Aab - Ba) + h\xi'(1 - 2Aa),$$

$$\frac{\partial \pi}{\partial \xi'} = h\xi(1 - 2Aa) - h(2Ab + B) - 2Ah^2\xi',$$

$$\frac{d}{dt}\frac{\partial \pi}{\partial \xi'} = h\xi'(1 - 2Aa) - 2Ah^2\xi'',$$

and therefore

$$\frac{d\Pi}{dx} = \int_0^{t_2}\{2Ah^2(p''(t) + x\psi''(t)) + 2a(1 - Aa)(p(t) + x\psi(t)) + (b - 2Aab - Ba)\}\psi(t)dt. \tag{4}$$

3. The Conditions (*a*) and (*b*). If (*a*) is to be satisfied with $\psi(t)$ arbitrary in its domain, we may deduce from equation (4), after putting $x = 0$, the vanishing of the multiplier of $\psi(t)$ in the integrand itself. Hence

$$2Ah^2p''(t) + 2a(1 - Aa)p(t) + (b - 2Aab - Ba) = 0. \tag{5}$$

In other words, $p(t)$ must be the continuous solution of this equation with continuous derivative which at $t = 0$ takes on the value p_1, and at $t = t_2$ the value p_2, if there is such a solution.

But in this case, for any x, $0 < x < 1$, the quantity $d\Pi/dx$ reduces to

$$\frac{d\Pi}{dx} = x\int_0^{t_2}\{2Ah^2\psi''(t) + 2a(1 - Aa)\psi(t)\}\psi(t)dt$$

$$= x[2Ah^2\psi'(t)\psi(t)]_0^{t_2} + x\int_0^{t_2}\{-2Ah^2(\psi'(t))^2 + 2a(1 - Aa)(\psi(t))^2\}dt,$$

as we partly retrace the integration by parts. This expression is in fact essentially negative, since the term outside the integral vanishes with $\psi(t)$ at 0 and t_2, and the integral itself is negative on account of the inequalities $A > 0$, $a < 0$.

Condition (*b*), as well as condition (*a*), is therefore satisfied if $p(t)$ is the

desired solution of (5). This equation is directly solvable in terms of exponential or hyperbolic functions. Thus if we introduce the constants

$$p_0 = \frac{b - 2Aab - Ba}{-2a(1 - Aa)}, \qquad m^2 = \frac{-a(1 - Aa)}{Ah^2},$$

the equation may be written in the form

$$p'' - m^2 p = -m^2 p_0, \tag{6}$$

and the general solution of the equation is seen by trial to be

$$p(t) = p_0 + C_1 e^{mt} + C_2 e^{-mt} \tag{7}$$

with C_1 and C_2 arbitrary constants and m the positive root of m^2. If we let $r_1 = p_1 - p_0$, $r_2 = p_2 - p_0$, substitution in this formula gives us

$$C_1 = \frac{r_1 - r_2 e^{mt_2}}{1 - e^{2mt_2}}, \qquad C_2 = \frac{r_1 - r_2 e^{-mt_2}}{1 - e^{-2mt_2}}. \tag{7'}$$

There is therefore always a unique solution of the problem proposed.

4. Discussion of Solution. A particular solution of (6) is the constant $p = p_0$. It is interesting to note that this price is the "Cournot monopoly price" obtained when the equation of demand does not involve dp/dt.[1] It continues to be a solution of our problem with $h \neq 0$ if the end values are properly chosen. Moreover the formulæ (7), (7′) show that the only solutions which remain finite as t_2 becomes infinite are those which approach p_0 asymptotically, *viz.*,

$$p = p_0 + r_1 e^{-mt}. \tag{8}$$

No solution of (6) which is not identically equal to p_0 can take on the value p_0 more than once; in fact, as is seen from (7), this will be the value of t, if that value is real, not negative and $\leq t_2$, for which

$$e^{2mt} = -C_2/C_1.$$

The calculation of the slope dp/dt gives an accurate picture of the graph of p as a function of t. From (7),

$$\frac{dp}{dt} = m[C_1 e^{mt} - C_2 e^{-mt}].$$

At the time $t = 0$, this has the value

$$p_1' = m(C_1 - C_2),$$

and the slope will vanish for the single value of t (if that value is real, $0 \leq t \leq t_2$) for which

$$e^{2mt} = C_2/C_1.$$

[1] See the author's paper, already cited.

By calculating the values of $C_1 - C_2$ and C_2/C_1 in terms of r_1 and r_2, the reader will find it interesting to verify the following facts with respect to the graph of price against time.

If r_1 and r_2 have opposite signs, the graph crosses the line $p = p_0$ once and has no horizontal tangent in the interval, the price continuously decreasing or increasing, as the case may be, from p_1 to p_2; if r_1 and r_2 have the same sign, the graph fails to cross the line $p = p_0$ at all, and has one and only one horizontal tangent provided the interval of time is large enough,—*i.e.*, provided

$$\cosh mt_2 \geq \frac{r_1}{r_2} \geq 1/\cosh mt_2.$$

Otherwise there is no horizontal tangent. Whether there is a maximum or a minimum is decided in any case of the existence of a horizontal tangent (since the second derivative does not vanish) by the slope of the graph at $t = 0$, and this is readily seen to have the same algebraic sign as the quantity

$$r_2 - r_1 \cosh mt_2.$$

But a comparison of this quantity with the previous inequality shows that when r_2 and r_1 are both positive the graph has a minimum, and when r_2 and r_1 are both negative the graph has a maximum; in other words, when r_1 and r_2 have the same sign, the graph is contained between p_0 and one of the values p_1 or p_2.

It is worthy of remark that these results are all independent of the sign of h in (1). It is merely a question of which end of the graph contributes most of the profit or least of the loss—since it is not essential that Π itself shall be positive.

5. An Extension of the Domain of $\psi(t)$. Obviously it is usually not possible to keep dp/dt continuous at the times $t = 0$ and $t = t_2$, as we see in the case when we change from one price p_1, which has been constant, to a second price p_2, which will be constant; it is therefore unreasonable to restrict ourselves to the consideration of curves whose derivatives remain continuous *within* the interval of time 0 to t_2. It is interesting to note that the solution already obtained is valid even when this restriction has been removed from the class of functions with which we deal in forming Π.

In fact, let $p(t)$ be the function already found, and $f(t)$ any other taking on the same initial and final values, continuous and with a derivative which remains bounded but may have discontinuities at a finite number of places.[1] Then $\psi(t) = f(t) - p(t)$ will be a function of the same sort. It remains to show that condition (6) will still be fulfilled.

If in the evaluation of the second member of (3) we separate out those terms which do not involve x, they will all cancel among themselves, for they form precisely the quantity $(d\Pi/dx)_{x=0}$ which vanishes as before on account of the

[1] More generally, let $f(t)$, taking on the same initial and final values as $p(t)$, be absolutely continuous and have a derivative which is summable, with its square.

choice of $p(t)$. Hence

$$\frac{d\Pi}{dx} = x\int_0^{t_2} [2a(1 - Aa)\{\psi(t)\}^2 - 2Ah^2\{\psi'(t)\}^2 + 2h(1 - 2Aa)\psi(t)\psi'(t)]dt.$$

But

$$2\int_0^{t_2} \psi(t)\psi'(t)dt = \int_0^{t_2} \frac{d}{dt}\{\psi(t)\}^2 dt = 0,$$

since $\psi(0) = \psi(t_2) = 0$, and the rest of $d\Pi/dx$ is essentially negative since $a < 0$, $A > 0$. The point is thus proved.

6. Related Problems. One purpose in writing the present paper, as well as the previous one, has been to show the wide range of problems suggested and solvable by a moderate mathematical equipment, and to encourage others to read in a direction that cannot but be fruitful.[1] The reader may for instance be tempted to apply the methods of this paper to the problem of competition discussed in the earlier article or, by means of a price index, commence the study of the changes in the demand and cost functions resulting from taxation. There are one or two specific questions that are worth mentioning because they lie close at hand.

What can be said about the extremals—that is, solutions of (6)—which start from a given value p_1 at $t = 0$? Which gives the greatest total profit during the arbitrary time t_2? With this form of the question the result depends upon the algebraic sign of h, the question not admitting an answer for all values of t_2 unless h is negative. But for very large changes in price it is not reasonable to assume the law (1) to remain valid. Instead of changing the law, however, it is desirable to limit the variation of price to some region $p_0 \pm M$ where M is some fixed amount. What can be said about the best price curve in this range, starting from p_1, as t_2 becomes indefinitely large?

A last question exhibits the ease with which one may be led into questions of considerable mathematical difficulty and interest. We can imagine as a further refinement to (1) a law in which the quantities a, h are not constants but functions of p, p'. As a likely hypothesis consider the case where a, b in (1) are given constants, but h has one of two values h_1, or h_2, according as p' is positive or negative respectively.

EULER'S OUTPUT, A HISTORICAL NOTE.

By W. W. R. BALL, Cambridge, England.

Professor D. E. Smith has enriched many of the recent numbers of the MONTHLY with extracts from letters in his possession on details of mathematical history. There is perhaps a similar interest in the following extract from a letter in my possession written by De Morgan and dated 17 October 1858. The first part of the extract only puts in a striking way what is familiar to many students;

[1] For example, the works of Cournot, Jevons, Walras, Pareto and Fisher. Those who can read Italian will find interesting the volume of Amoroso, already cited.

the allusion in it to 1000 miles in 1000 hours refers of course to the famous bet, on the issue of which about £100,000 was staked, made by Captain Barclay Allardyce in 1809 that he would walk one mile in each of a thousand consecutive hours, covering nearly six weeks. The story in the second part of the extract rests I believe only on tradition, but I have heard it from other sources and have no reason to doubt its truth.

Extract from De Morgan's Letter.—Euler's life, dating from 1736, the year in which his productions began to appear with rapidity, is a period of 47 years; during the last 17 years he was totally blind, and throughout the whole of it he suffered from the consequences of a fever which deprived him of the sight of one eye at its commencement. Nor was he secluded from the world: he married a second wife, and was the father of 13 children. His life was not free from such calamities as interrupt the course of study. He saw the deathbeds of ten of his children; his house was set on fire and wholly burnt; and an attempt to restore his sight by couching led to an illness which nearly ended his days. He was fond of conversation, of the society of his family, and of music; and was, throughout the whole of his career, at the orders of a royal or imperial patron. Nevertheless, if his memoirs be counted, and if his separate works (not volumes) be allowed for at the average rate of 20 memoirs each, which is an insufficient rating both as to bulk and matter, the result is as follows: Distribute Euler's work through the whole period—the real and average distributions will not much disagree—and there is for each and every fortnight in 47 years a separate effort of mathematical invention, digested, arranged, written in Latin, and amplified, often to a tedious extent, by corollaries and scholia. Through all this mass, the power of the inventor is almost uniformly distributed, and apparently without effort. There is nothing like this, except this, in the history of science: it is the thousand miles in the thousand hours.

When Euler was at Berlin, and there is no reason to suppose it was otherwise at St. Petersburg, he was in the habit of writing memoir after memoir, and placing each, when finished, at the top of a pile of manuscript. The secretaries of the academy helped themselves from time to time, by taking papers from the top of the pile, according to their estimate of the bulk of matter likely to be wanted for reading. The consequence was that, as the pile often increased more rapidly than the demands upon it, the memoirs which happened to be at the bottom remained there for a long time. This explains how various memoirs of Euler were published, though considerable extensions and improvements of the matter contained in them had been previously published [under his name].

QUESTIONS AND DISCUSSIONS.

EDITED BY C. F. GUMMER, Queen's University, Kingston, Ont., Canada.

The department of Questions and Discussions in the Monthly is open to all forms of activity in collegiate mathematics, including the teaching of mathematics, except for specific problems, especially new problems, which are reserved for the separate department of Problems and Solutions.

QUESTIONS.

The following question is suggested by Professor Dadourian's note on the catenary, which is printed below.

51. Can any reader supply approximate formulas for the problem of a cable suspended from two points at different levels?

DISCUSSIONS.

I. NOTE ON THE CATENARY.

By H. M. DADOURIAN, Trinity College.

The object of this note is to derive a number of exact and approximate relations which might be of use in connection with uniform and flexible cables suspended from two points.

Let s denote the length of that portion of the cable which lies between its lowest point and any of its points $P(x, y)$, T be the tension of the cable at $P(x, y)$, T_0 the tension at the lowest point of the cable and w the weight of the cable per unit length. Then it may be shown that the following relations hold:[1]

$$T^2 = T_0^2 + w^2 s^2, \tag{1}$$

$$s^2 = y^2 - a^2 \qquad (a = T_0/w), \tag{2}$$

$$2s = a(e^{x/a} - e^{-x/a}). \tag{3}$$

Let $2D$, $2L$, H, and T_m denote, respectively, the span, the length, the dip below the suspension points, and the maximum tension of the cable. Then, for $s = L$, $y = H + a$. Making these substitutions in (2) we have

$$a = (L^2 - H^2)/2H; \tag{4}$$

and consequently

$$T_0 = (L^2 - H^2)w/2H. \tag{5}$$

Introducing this value of T_0 into (1) and simplifying, we obtain

$$T_m = (L^2 + H^2)w/2H. \tag{6}$$

From (5) and (6) we get

$$L = (\sqrt{T_m^2 - T_0^2})/w. \tag{7}$$

When the tension of the cable is very great; in other words, when the sag is very small, simple expressions may be obtained for L, T and T_m in terms of the easily measurable magnitudes D, H and w.

[1] *Cf.* Dadourian, *Analytical Mechanics*, 2nd ed., p. 109.

Expanding the right-hand member of (3) in powers of x, we have

$$s = x + x^3/(a^2 3!) + x^5/(a^4 5!) + \cdots. \tag{8}$$

Therefore we obtain $L = D$ for a first approximation and $L = D + D^3/6a^2$, or $L = D[1 + 2D^2H^2/3(L^2 - H^2)^2]$, for a second approximation. In the correction term of the last equation $L^2 - H^2$ may be replaced by D^2 and the following relation obtained:

$$L = D(1 + 2H^2/3D^2). \tag{9}$$

Thus the increase in length due to sagging is given by

$$\Delta L = 2H^2/3D; \tag{10}$$

that is,

$$\Delta L = D^3w^2/6T_m^2. \tag{11}$$

From (5), (6) and (9) we obtain, after neglecting terms containing H^4/D^4,

$$T_0 = (D^2 + H^2/3)w/2H, \tag{12}$$
$$T_m = (D^2 + 7H^2/3)w/2H.^1 \tag{13}$$

II. An Inequality in Connection with Logarithms.

By A. A. Bennett, University of Texas.

Two of the most obvious questions that present themselves in an introductory logical study of logarithms are: (1) How can one demonstrate that the tables of common logarithms are correct (to the given number of places) even in such simple cases as for instance $\log_{10} 2$ and $\log_{10} 3$? (2) How is one to be convinced in a few steps that the limit of $(1 + x/n)^n$ as n tends toward infinity through positive integral values, exists, and can indeed serve as a definition of e^x (confining ourselves to real values of x), without recourse to the theory of infinite series?

It is of some interest that a single elementary inequality can be used to answer both of these questions and others related to them. Thus two problems arising from the study of a common subject but having otherwise apparently little in common are again related by a common origin for the proof, although this common element is put to different use in the two cases. The inequality is the following,

$$(1 + x/m)^m(1 - x/n)^n < 1, \tag{A}$$

for every real x and every pair of natural numbers m and n not less than the numerical value of x.

This inequality is readily established by induction. Indeed we have, in the first place, $(a - d)/a < a/(a + d)$ for $a > d > 0$, since on clearing of fractions,

[1] The circle of curvature at the lowest point of the catenary has a radius a, and the equations (2), (3) and (8) apply to this circle to the order of approximation used in this paper. Hence also (10) is approximately true when the circle is substituted for the catenary. Consequently graphical methods based on the assumption of a circular arc are admissible to the desired degree of accuracy. The formulas (2) and (4) are of course exact for the catenary, but not for the circle.
Editor.

a and $a + d$ being positive, $a^2 - d^2 < a^2$. Similarly, if we write $a - d$ in place of a, $(a - 2d)/(a - d) < (a - d)/a$ for $a > 2d > 0$. Multiplying these two inequalities of like sign together we have *a fortiori* $[(a - 2d)/(a - d)][(a - d)/a] < [a/(a + d)]^2$, if $a > 2d > 0$. More generally, $(a - md)/a < [a/(a + d)]^m$. Hence $[(a - md)/a]^n < [a/(a + d)]^{mn}$. Continuing in this manner, we obtain

$$[(a - md)/a]^n < [a/(a + nd)]^m, \tag{B}$$

for $a > md > 0$. A slightly more general inequality follows at once from this, namely,

$$[(a - md)/a]^n < [(a + c)/(a + c + nd)]^m, \tag{C}$$

for $a > md > 0$, and $c > 0$. Writing ax/mn in place of d, we have, on condition that $n > x > 0$, the desired inequality (A) for positive real values of x. From the symmetry of the equation, we conclude its validity also for x negative.

Let us first take up the second of the two questions proposed. For convenience in writing we shall use the symbol $E(x, m)$ to denote the expression $(1 + x/m)^m$. We shall also break up the discussion into a succession of numbered propositions, some of which are little more than lemmas. We shall assume that $E(x, m)$ is defined for all real finite values of x and m for which $(1 + x/m) > 0$. This assumes familiarity with the concept of fractional, irrational, and negative exponents as introduced in the subject of elementary algebra.

1. $E(x, m)$, for $m > 0$, is continuous in x and m. The details of the proof will be omitted.

2. For p, q, real numbers for which $1 + qx/p > 0$, $[E(x, p/q)]^q = E(qx, p)$. An analytical identity.

3. For a given positive m and for x ranging over positive values, $E(x, m)$ is an increasing function of x. The proof is obvious if the positive mth root be taken.

4. For a given positive x, and for m ranging over positive values, $E(x, m)$ is an increasing function of m. The proof will be carried through several steps. For positive integral values of m, $E(x, m)$ may be written as a polynomial in x of the form

$$1 + x + (1 - 1/m)x^2/2! + \cdots + (1 - 1/m)(1 - 2/m)\cdots(1/m)x^m/m!.$$

The result of replacing m by $m + 1$ is to increase each of the $m + 1$ terms already secured and to add a new final term which is positive. The theorem is therefore established for positive integral values of m. To compare the values of $E(x, m)$ for two positive rational values of m it is sufficient that we reduce these to a common denominator, and compare the results for the two numerators. In other words, to prove the proposition for rational numbers it will suffice to show that for each fixed positive integral denominator, q, $E(x, p/q)$ is an increasing function of the numerator regarded as ranging over natural numbers. Proposition 2 above serves to prove this case. In view of the continuity mentioned in 1 above, the complete proposition follows.

5. $E(x, m)$ is bounded for any real positive x, as m increases through positive real values. Indeed from (A) we have an infinite set of inequalities, namely, $E(x, m) < 1/E(-x, n)$ for every natural number, n, greater than x. This holds at once if m be a natural number, but by 4 continues to be true for all real positive values of m.

6. The limit of $E(x, m)$ for x given and m approaching infinity exists for every positive real x. Let this be denoted by $E(x)$.

7. If $0 < x < y$, $E(x) < E(y)$.

8. $[E(-x, m)]^{-1} = E(x, m-x)\cdot[1 + x/(m-x)]^x$ for $m > x > 0$. An analytic identity.

9. The limit of $E(-x, m)$ for x given and m approaching infinity exists for every positive real x. Let this be denoted by $E(-x)$. It is sufficient to pass to the limit in each factor of the right-hand member of 8, and to note that neither limit is zero. Since the second factor approaches unity and the first factor approaches $E(x)$, we have

10. $E(-x) = 1/E(x)$.

11. $E(ax) = [E(x)]^a$ for real values of a. We have, from 2, $E(x, m)^a = E(ax, am)$. For a positive, we may let m increase toward infinity and thus prove the proposition in this case. For a negative, we may write $a = -b$, and use 10 above, and the result just obtained for b in place of a.

12. $E(x + y) = E(x)\cdot E(y)$. This follows at once from 11, by writing $y = cx$.

13. $[(m + x)/m]^m \leq E(x) \leq [n/(n - x)]^n$ for every positive m and n for which also $m + x$ and $n - x$ are positive. For x, zero, it is evident that the equality sign must be used throughout. For x, positive, the inequality for the first two members is a restatement of 4 above, after the notions of 5 and 6 are introduced. The inequality between the first and third members for x positive, is a consequence of (A). The above reasons also show that at worst there could be only an equality between the second and third members for certain choices of x and n. For x negative, we may write $y = -x$, and express the relations in terms of y. Inverting, rearranging and making use of 10, we have from the relation, $[(n + y)/n]^n < E(y) \leq [m/(m - y)]^m$, established for y positive, the relation desired with an inequality between the second and third members. We shall not spend the time in showing that the equality holds only for $x = 0$.

The relation, 13, for the case in which $m = n = 1$, is well-known.

The propositions given above not only serve to show that $E(x)$ as here defined may be written in the form of e^x, where $e = E(1)$, but serve to give simple approximations to e^x, even for large values of x, which is more than can be said for the series, owing to the slow convergence of the latter, although of course modifications may be introduced. The arithmetical character of the present treatment is of course its chief interest.

Turning now to the first of the two questions proposed, we shall find it more convenient to use the fundamental inequality in the form (B). Being interested in common logarithms, and in particular in the logarithms of 2 and of 3, we may make use of powers and products of these, as also like expressions involving

10/2, namely 5. The interesting feature of the inequality is that if several available numbers are given near to each other, new expressions are obtained numerically much larger but with ratios more nearly equal to unity. Thus from 4, 3, 2, we have $4/3 < 3/2$, whence $8 < 9$. From $8/9 < 9/10$, we get $80 < 81$. Using the more general form, (C), we have, from $25/24 < 16/15$, the relation $125 < 128$. This one relation when written in the form $1000 < 1024$ suffices to show that the logarithm of 2 is slightly in excess of 0.3. From $(9/10)^2 < 10/12$ we have $243/250 < 1$.

Without going further in this direction, we shall see what we can obtain by use of the ratios, 80/81, 125/128 and 243/250, alone, and their products. To simplify the problem, we shall consider merely products not involving more than two of these fractions as factors. Testing these for relative magnitude, and arranging accordingly, we have the following in increasing order of magnitude:

$$1,\ (250/243)\cdot(125/128),\ (128/125)\cdot(80/81),\ 81/80,\ 128/125,\ 250/243.$$

Multiplying by the common denominators of these taken three in succession at a time, we have the following products to consider:

$$\begin{aligned}
2^6 \times 3^5 \times 5^2 &= 388800,\\
5^8 &= 390625,\\
2^{17} \times 3 &= 393216,\\
2^2 \times 3^9 \times 5 &= 393660.\\
2^{15} \times 5 &= 163840,\\
3^8 \times 5^2 &= 164025,\\
2^{11} \times 3^4 &= 165888.\\
3^9 \times 5^2 &= 492075,\\
2^{11} \times 3^5 &= 497664,\\
2^5 \times 5^6 &= 500000.
\end{aligned}$$

Applying (B) to these, we have

$$\left(\frac{2^6 \times 3^5 \times 5^2}{5^8}\right)^{2591} < \left(\frac{5^8}{2^{17} \times 3}\right)^{1825},$$

$$\left(\frac{5^8}{2^{17} \times 3}\right)^{444} < \left(\frac{2^{17} \times 3}{2^2 \times 3^9 \times 5}\right)^{2591},$$

$$\left(\frac{2^{15} \times 5}{3^8 \times 5^2}\right)^{1863} < \left(\frac{3^8 \times 5^2}{2^{11} \times 3^4}\right)^{185},$$

$$\left(\frac{3^9 \times 5^2}{2^{11} \times 3^5}\right)^{2336} < \left(\frac{2^{11} \times 3^5}{2^5 \times 5^6}\right)^{5589}.$$

Taking logarithms of both sides, collecting terms, and writing x in place of log 2, and y in place of log 3, and hence $1 - x$ in place of log 5, we have the following

four inequalities:

$$\begin{aligned} 76717x + 14780y &< 30146, \\ -52556x + 20284y &< -6143, \\ 32213x - 15644y &< 2233, \\ -97436x - 18601y &< -38206. \end{aligned}$$

Unlike the case with equalities, all four of these relations are required in order to provide suitable inequalities for x and y. Combining we obtain the desired relations in the form

$$\begin{aligned} .301029 + &< \log_{10} 2 < .3010301 -, \\ .47712 \ \ - &< \log_{10} 3 < .47712 +. \end{aligned}$$

In this work there has been no emphasis upon peculiar properties not available for other numbers, beyond utilizing the fact that 2 is a factor of 10. The work is made more complicated but at the same time gives much closer approximations as more independent prime numbers are considered. Had we confined ourselves to 2 and 5, we would have had only such fractions as 4/5, 5/8, 125/128, 15625/16384, 1953125/2097152, etc., while by introducing 7 in addition to 3, we would have had even with small numbers all of the following with which to commence, 2/3, 3/4, 4/5, 5/6, 6/7, 7/8, 8/9, 9/10, 14/15, 15/16, 20/21, 24/25, 27/28, 32/35, 35/36, 48/49, 49/50, 63/64, 80/81.

These methods may be compared with those given in such algebra books as discuss the direct computation of common logarithms from first principles.

For a different treatment[1] of some of the questions in the first part of this article, see M. B. Porter, "The derivative of the logarithm," this MONTHLY (*1916*, 204–206). Some remarkable approximations are obtained in a systematic and very ingenious method by E. B. Escott: "The calculation of logarithms," *Quarterly Journal of Pure and Applied Mathematics* (1910) pp. 157–167; numerous references are given.

III. A MATHEMATICAL TREATMENT OF PERSPECTIVE.

By J. P. BALLANTINE, Columbia University.

The purpose of this paper is to demonstrate that a picture can be observed and appear exactly like the object only if observed from a single point, called the point of vision, and to give a construction for the point of vision in terms of the three vanishing points for three mutually perpendicular sets of parallel lines.

The author hesitates to submit this paper since the theory is so simple as to be almost trivial, but still the matter does not seem to be of general knowledge, since it is possible to find in mathematical text-books of the highest grade reproductions of photographs whose point of vision is from one to two inches from the printed page.

[1] A treatment of the exponential function based on algebraic inequalities, using no infinite series, and having some resemblance to the method forming the first part of the present paper, may be found in G. Chrystal, *Algebra*, Part II, Edinburgh, 1900, pp. 77–80. EDITOR.

Postulate. A plane P contains a drawing of an object in correct perspective as observed from a point A whose position is fixed with regard to the plane, if and only if the object can be so placed that the projection of the object on P from A coincides with the drawing.

Definition. If a set of parallel lines is projected on a drawing plane, the point of intersection of the projections is called the vanishing point for the set of lines.

Theorem. The direction from the point of vision of a drawing to the vanishing point of a set of parallel lines is parallel to the lines.

Theorem. If A and B are vanishing points for two sets of parallel lines making the angle α, then A and B subtend an angle α at the point of vision.

Theorem. If A and B are vanishing points for two sets of parallel lines which are mutually perpendicular, then the point of vision lies on a sphere of which the line AB is a diameter.

Theorem. If A, B, and C are the three vanishing points corresponding to three mutually perpendicular sets of parallel lines, then the point of vision lies at a point of intersection of the three spheres with lines AB, AC, and BC as diameters, and is thereby uniquely determined as a single point in front of the drawing.

If we consider a printed photograph of a building in which three sets of mutually perpendicular lines appear, the three vanishing points can be constructed, and it is surprising to notice how often the point of vision is much closer to the picture than is convenient to observe from. In the case of many drawings of buildings where a strict observance of the rule of two-point perspective as commonly used by artists, in which vertical lines are supposed to be drawn parallel, would make the two upper edges of the building appear to come together in an acute angle, the artist realizes the error and fudges the straight lines into curves so as to make the angle obtuse. Such a practice is an overt admission that the theory used is only an approximation.

RECENT PUBLICATIONS.

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REVIEWS.

The Reorganization of Mathematics in Secondary Education. A report by the National Committee on Mathematical Requirements. Published by the Mathematical Association of America, 1923. x + 652 pages.

For many months after the appearance of the preliminary report of The National Committee on Mathematical Requirements, teachers of mathematics throughout the United States waited more or less patiently for the promised complete report. The fact that the members of this committee are mathematicians and teachers whose reputations are nation-wide led us to look for a valuable report. In its final form it far exceeds all expectations. The possi-

bilities of a committee of this sort, adequately financed, working over a period of years, are fully exemplified in the publication now under discussion.

It consists of six hundred fifty pages besides valuable plates, and is divided into two principal parts. The first part deals with "General principles and recommendations." These are committee reports divided into eight chapters. The second part consists of eight chapters giving reports on special investigations conducted by various individuals for the committee. The first chapter is merely a "bird's-eye view" of the entire report. The real work of the committee begins in the second chapter.

In these days, when state legislation and survey reports are opposing required mathematics in secondary schools, it is highly essential that arguments be presented in justification of the retention of these courses in the high school curriculum. Pupils and parents are constantly asking the question "What's the stuff good for?" It is incumbent upon the teachers of mathematics in colleges and secondary schools satisfactorily to answer this question. This report furnishes ample material for meeting these demands. The second chapter in particular is devoted to a discussion of the valid aims and purposes of instruction in mathematics. These aims are divided into practical, disciplinary, and cultural, each having several sub-heads. One sentence under the topic "The point of view governing instruction" gives the kernel of the entire chapter: "The primary purposes of the teaching of mathematics should be to develop those powers of understanding and of analyzing relations of quantity and of space which are necessary to an insight into and control over our environment, and to an appreciation of the progress of civilization in its various aspects, and to develop those habits of thought and of action which will make these powers effective in the life of the individual." The subject of unified mathematics is given a very brief but fair treatment under the heading "The organization of subject matter." The necessity of thorough professional training for teachers is presented in the latter part of the chapter.

In chapters three and four, the committee outlines the work in mathematics to be covered in grades seven to twelve inclusive. The introduction in grades seven to nine of such topics as line, bar, and circle graphs; statistical graphs; simple frequency distributions; intuitive geometry; and numerical trigonometry may at first cause somewhat of a shock to some of our more conservative educators. However, if the chapters and references are carefully studied, the proposed plan will not appear as revolutionary as it may seem at first sight. The writers of these chapters plainly state that they are not able in all cases to advise as to the order of topics recommended, and thus very wisely avoid committing themselves on the subject of unified mathematics. In discussing the work of the last three years, they make the following statement: "In the majority of high schools, at the present time, the topics suggested can probably be given most advantageously in separate units of a three year program. However, the National Committee is of the opinion that methods of organization are being experimentally perfected whereby teachers will be enabled to present much of

this material more effectively in combined courses unified by one or more of such central ideas as functionality and graphic representation." The striking features of the fourth chapter are the arguments in favor of taking the elective courses suggested for this period, and especially for the presentation of elementary calculus.

In the discussion of "College entrance requirements" the attitude of the committee is unusually fair and unbiased. The position is taken that the high school courses in mathematics should prepare for college courses in general, not mathematics courses only. With this in view, a tabulation is given showing the findings of a committee that endeavored to determine what topics are essential for other college courses. It is furthermore conceded that since college entrance requirements exert such a marked influence upon secondary school teaching, they must "reflect the spirit of sound progressive tendencies." The college entrance examinations, so influential in the eastern section of the United States, are criticized in a very able manner, and valuable suggestions are given for their improvement. Detailed lists of minor requirements and major requirements in algebra are carefully worked out. In the following chapter, the essential propositions of plane and solid geometry are tabulated.

Perhaps the most valuable article in the entire report in actually helping teachers to improve their methods of presenting secondary school mathematics is that on "The function concept in secondary school mathematics." Under this topic E. R. Hedrick brings out in a very simple manner the possibilities of showing the points of contact between algebra, geometry, trigonometry and the problems of every-day experience. Numerous illustrations of the interrelations between quantities are brought to the attention of the reader.

The efforts of the committee in the last chapter of part one to establish greater uniformity in the use of terms and symbols are highly commendable. In every case, the aim seems to be to substitute the simple, commonsense term for the more difficult or technical one. Symbols not universally used are not recommended. Every writer of examination questions, text books in mathematics, or contributions to educational journals would do well to read this chapter.

The first article in the second part of the report deals with the old subject of formal discipline. From a study of twenty-nine important experiments in "transfer," Miss Vevia Blair drew up a set of seven apparently logical conclusions. These were very carefully worded and submitted to forty prominent educational psychologists for their approval or disapproval. One psychologist accused Miss Blair of asking suggestive questions, and so framing them that the great majority of the answers must necessarily be in the affirmative. Whether or not this charge is justifiable, Miss Blair is to be congratulated on getting twenty-four of these forty to state their views in plain English on this much discussed question. The replies are exceedingly interesting, and definitely indicate a general belief in a certain degree of transfer of training. Within certain limitations, the transfer of training is held to be a valid aim in teaching, the amount of transfer being largely dependent upon the methods of teaching.

"The theory of correlation applied to school grades" by A. R. Crathorne is a masterful illustration of the possibilities of the modern theory of statistics applied to problems of secondary education. The article is equally valuable to teachers of mathematics and to those interested in statistics in showing practical applications of the correlation coefficient, correlation ratio, partial correlation, and regression equations. The conclusions are indicative of some important relationships between the various high school subjects, and seem to be entirely consistent with the results of other investigations in the same field.

Although the chapter by J. C. Brown on "Mathematical curricula in foreign countries" deals with conditions as they were previous to the war, it is evident that in many respects we might well follow the lead of our European neighbors in secondary school mathematics. Mr. Brown discusses the curricula in thirteen European countries. He then summarizes the work by years, and makes a comparison by graphs and by tabulations. After reading this chapter, one is chagrined at the superior preparation of the European teacher of secondary mathematics. It is evident that greater emphasis is placed upon thoroughness through drill. Both algebra and geometry are introduced much earlier in the course than is the case in this country, and there is a much closer correlation between the subjects. Two sentences quoted from Mr. Brown show clearly his attitude: "European school men believe that a course in mathematics should be planned by those who know some mathematics rather than by educators who are practically ignorant of the subject. The reports do not indicate that the schools of Europe are hearing a demand for weak algebra and anæmic geometry, or even for no work in these subjects."

"Experimental courses in secondary school mathematics"—a report by Raleigh Schorling—occupies slightly over one hundred pages. Owing to the fact that the junior high school is still in the formative period, most of the experimentation is confined to the seventh, eighth, and ninth grades. A detailed description is given of at least one course in each of fifteen prominent schools. In every case, the school has made a distinct contribution under the leadership of one especially trained in mathematics and in education. The headings of some of these reports will best indicate the comprehensiveness of this study: What mathematics is needed by industry?—Correlated mathematics—What mathematics is demanded by commerce?—An early effort to construct a course with little reference to later courses—The problem of plane geometry—A program of coöperative research—No mathematics in the ninth grade—What mathematics should a boy study who can stay in school only ten weeks?—The project method —Progressive correlation. These accounts of what has actually been done to answer such questions as the above constitute the third part of the chapter, and furnish the source material for the conclusions which are summarized in part one and discussed and illustrated in part two. The conclusions are in accord with the recommendations of chapter three. This chapter merely strengthens the former chapter by showing that such courses have actually been offered successfully.

In accordance with the present popularity of the testing movement, the subject of "Standardized tests in mathematics for secondary schools" is given by far the most space of any subject. This report by C. B. Upton occupies one hundred and fifty pages, or nearly one-fourth of the entire publication. It gives a rather minute description of the various tests in arithmetic, algebra, and geometry with an explanation of the methods of using them. In addition to the standardized educational tests in these three subjects, the Rogers test of mathematical ability, and the Thurstone vocational guidance tests are discussed. The chapter closes with a valuable bibliography of the subject. The only logical conclusion to be drawn from this article is that when a subject as new as this (not yet ten years old) has been accorded such an important position by a committee of mathematicians—not an over-zealous group of educational specialists in measurements—it behooves every mathematician in the land to join the procession. Two sentences from the author's conclusion summarize the importance of tests: "They have begun the detailed analysis of mathematical abilities, they have rendered possible improved methods of classification, and diagnosis and prognosis of individual ability can now be made by this help. Analysis, classification, and diagnosis—these three outstanding ends they serve; and the greatest of these is diagnosis."

In chapter fourteen, we are made to feel even more keenly than in chapter eleven the inferiority of our mathematics courses as compared with those of European countries. This chapter is entitled "The training of teachers of mathematics," and is written by R. C. Archibald. The first part is a summary of a more extensive report published by the Bureau of Education in 1918, dealing with the training of teachers of mathematics in foreign countries. The greater part of the chapter is devoted to a summary of the requirements in the various sections of the United States for the certification of teachers, and of an outline of the courses offered by the colleges and universities of these sections. The chapter closes with a proposed tentative ideal for the preparation of teachers which deserves high commendation, but is decidedly too ambitious in its aims to permit hope for realization in the near future.

Reports of investigations by the questionnaire method constitute the next chapter. The first one is by A. R. Crathorne on "Change of mind between high school and college as to life work." About two thousand freshmen from eleven different colleges answered questions indicating changes in their plans. Interesting tabulations of the various answers are given and summaries are made. The figures show that approximately 57 per cent. had definite plans when they entered high school, and that about half of these changed their plans.

An investigation on "Mathematical interests of high school pupils" was conducted by W. F. Downey. Seven thousand pupils from high schools in the east, west, mid-west, and southwest answered questions concerning their likes and dislikes in mathematics, and in other subjects. The summary shows that four-fifths of the pupils like mathematics, and that the popularity of mathematics compares well with that of other subjects.

Mr. Alfred Davis under the subject "The importance of mathematics as indicated by certain questionnaires" tabulates the results of questions sent out by himself and others to citizens and pupils concerning their attitude toward mathematics. Some of these have been reported elsewhere, but none of these reports leave any doubt in our minds as to the favorable attitude of business men, professional men, and pupils toward the study of mathematics.

A bibliography of nearly one hundred pages covering every possible phase of the teaching of mathematics constitutes the last chapter of the report. This alone is of inestimable value to every teacher of mathematics.

Too much cannot be said in commendation of the National Committee for producing such a comprehensive report, and of the Mathematical Association of America for sponsoring it. It should most certainly be upon the desk of every teacher of secondary mathematics, and should be used in every university and college class dealing with the problems of mathematics teaching. The effects of this work in increasing interest in mathematics and improving the quality of teaching cannot be over-estimated.

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Algebras and Their Arithmetics. By L. E. DICKSON. The University of Chicago Science Series. The University of Chicago Press, 1923. 8vo. xii + 238 pages. Price, $2.35, postpaid.

"The chief purpose of this book is the development for the first time of a general theory of the arithmetics of algebras, which furnishes a direct generalization of the classic theory of algebraic numbers. . . . Just as the final stage in the evolution of number was reached with the introduction of hypercomplex numbers (which make up a linear algebra), so also in arithmetic, which began with integers and was greatly enriched by the introduction of integral algebraic numbers, the final stage of its development is reached in the present new theory of arithmetics of linear algebras."

According to this extract from the preface, the book is primarily an exposition of the author's research, and must therefore be judged mainly according to the novelty and importance of his results. Perhaps the fairest way to form an estimate of the book is to compare the results of the author with those obtained by previous investigators in this field.

It often happens in extending an elementary concept to fit a more general situation that different investigators devise different generalizations. Each of these extensions may be a true generalization of the elementary concept and hence entitled to the same name. But in such cases it is usually true that one definition allows more theorems to carry over from the elementary situation than does any of the others, and is therefore by common consent taken as the standard definition.

This is the situation with regard to the generalization of the term *integer* in passing to linear associative algebras over a general field. A. Hurwitz and Du Pasquier have proposed definitions for the integral elements of linear algebras, and have done considerable work in investigating the consequent theorems for certain algebras. These theorems proved to be decidedly meagre, and indeed their definitions left many algebras entirely without integral elements.

Professor Dickson has recently proved that for a certain algebra the integers of Du Pasquier not only lack the property of unique factorability, but that unique factorability cannot be restored by the introduction of ideals no matter how defined.

In the present work Professor Dickson solves the problem of finding a satisfactory definition for the integral elements of a linear associative algebra, and develops many general properties of them in a manner which makes the reader marvel at his genius for clear and profound analysis. He expresses an algebraic field as a linear algebra, and makes a careful study of its integral elements. By a shrewd generalization he is led to his definition for the integral elements of any algebra. The definition seems to be a happy one, for many interesting general theorems flow easily from it, in marked contrast to the experience with the earlier definitions. All algebras which have integral elements according to Du Pasquier's definition continue to have them under Dickson's definition while the converse is not true. There can be no doubt that the book under review opens up a new field for research which is of vast extent.

The new theory has its practical side also, for applications are made to the solution of Diophantine equations. Only one such equation is solved in detail, but the method is shown to be very powerful.

The first chapter is an introduction to the theory of linear algebras. Attention is called to the fact that the set of all p-rowed square matrices is a linear algebra in p^2 units. An interesting paragraph contains the derivation of quaternions as a special case of a matric algebra over the complex field. The theory of linear dependence with respect to a field is also developed in this chapter.

Chapters II to VIII inclusive are devoted to an exposition of the theory of linear algebras over a general field, the theory due essentially to Wedderburn. This is the most complete and satisfactory theory of linear algebras that has yet been formulated, and the present book is the only place where the theory is to be found outside of foreign mathematical journals. Several simplifications are published here for the first time. In fact, the author communicated freely with Professor Wedderburn during the writing of the book. The more recent work of Scorza was also consulted. As a text on linear algebras, therefore, this book is without rival.

Chapter IX is a concise development of the fundamentals of algebraic number theory. Chapter X, which is 59 pages in length, contains the author's theory of the arithmetics of linear algebras, and has already been discussed.

The concluding chapter XI is on the foundations of algebra, and makes interesting reading. A set of postulates for a field is given and several fundamental theorems are derived from them. The theory of indeterminates and polynomials over a field is developed, and the usual theorems on divisibility of polynomials are proved with a minimum of effort. The chapter closes with a discussion of the congruential theory of polynomials, and Galois fields.

In spite of the inherent difficulty of the subject matter, the book has been written with a view of interesting a wide class of readers, and special effort has

been expended to make the presentation clear and elementary. Only a good general knowledge of mathematics is required, and the preliminary chapters are designed to prepare for an understanding of chapter X. The subject matter is so skillfully arranged and so clearly presented that the book is easy to read and to understand. The use of examples to illustrate the difficult parts is also helpful.

It must not be assumed however that the book is an afternoon's light reading. The author once stated that he did not expect anyone to read his books without a paper and pencil, and the reader will do well to bear this in mind The steps are all given and the proofs are accurate, but there is no exhausting detail. Thus in the middle of page 139 we have a situation which may worry some readers, for there is what appears to be an infinite series $c\omega_2 + c'\omega_2' + \cdots$ with no mention of convergence. A little use of the pencil however will show that all but a finite number of the c's are zero.

In the first two appendices are to be found two rather complicated proofs, and in Appendix III a summarized statement of further results on linear algebras and full references to the original papers in the literature. A unique feature is the statement in this appendix of a number of unsolved problems, practically an invitation to the reader to try his hand at research in this field. The enthusiasm of the author for mathematical research is so great that the reader of this book can hardly help feeling inspired by it, and ambitious to do something on his own account.

The book is well printed with attractive type and thus speaks well for the University of Chicago Science Series of which it is the fifteenth number, and for the Editorial Committee of the Series, of which Professor E. H. Moore is chairman.

Perhaps it is not out of place to state that the mathematical world is indebted for this book not only to the author but also to the University of Chicago Press. Needless to say, such a book as this is financially a losing venture and only the generosity of the Press has made its publication possible. But we must not take this generosity too much for granted, for if more books of this caliber are to be printed by our university presses they must receive the financial as well as the moral support of the public.

C. C. MacDuffee.

Lezioni di Geometria Analytica. By E. Bortolotti. Bologna, Nicola Zanichelli, 1923. 8vo. Vol. 1, xxxix + 382 pages. Vol. 2, 229 pages.

This book contains many points of interest to American readers. An introduction of thirty pages gives an unusually full historical development of analytical and projective geometry, covering such topics as the geometric algebra of the ancients, the Greek construction of conics by points, the classic work of Apollonius, the decline of Greek geometry and its renaissance in Italy many centuries later, the researches of Desargues, Torricelli, Descartes and Fermat. A student reading this historical exposition will fully recognize as mythical the statement current in some quarters that the magic pen of Descartes brought forth analytical geometry—*prolem sine matre creatam.*

Conspicuous is the total absence of problems for the student to solve. But in the preface the author stresses the importance of problems and states that he reserves them for publication in a separate volume.

A striking feature of the book is the discussion, in the very first chapter, of imaginary elements, of baricentric coördinates, projective coördinates and homogeneous coördinates. The treatment of the point and straight line is given in the second chapter for plane geometry and in the third chapter for that of space. In the latter chapter are considered also (1) the plane, (2) imaginary points, planes and lines in space, (3) the plückerian or tangential coördinates of planes, as well as the homogeneous cartesian and the general projective coördinates of points and planes. A knowledge of determinants is presupposed.

The second part of the book deals with conics. By this time the student at the University of Bologna has become familiar with limits and derivatives in other mathematical courses and can use that knowledge here. The general treatment of the equation of the second degree in x and y is followed by the consideration of the normal equations of the ellipse, hyperbola and parabola. An appendix deals with homography and reciprocity.

The second volume contains the third and fourth parts which deal, respectively, with quadric surfaces and with the elements of the general theory of curves and surfaces.

The presentation of analytical geometry as given here possesses many points of elegance. The divergence from American texts in the selection and the sequence of topics renders Bortolotti's book of special interest to us.

FLORIAN CAJORI.

ARTICLES IN CURRENT PERIODICALS.

ANNALES DE L'ÉCOLE NORMALE SUPÉRIEURE, volume 58, no. 10, October, 1923: "Analogie entre les séries trigonométriques et les séries sphériques au point de vue de leur sommabilité par les moyennes arithmétiques" (continuation) by E. Kogbetliantz, 289–320.

BULLETIN DES SCIENCES MATHÉMATIQUES, second series, volume 47, September, 1923: Review by E. Cartan of T. Levi-Civita and U. Amaldi, "*Lezioni di Meccanica Razionale. Vol. 1: Cinematica principi e statica.*" (Bologne, 1923), 289–290; "Discours sur Marc Seguin, prononcé à Annonay le 10 Juillet 1923" by E. Picard, 291–298; "Le problème de Dirichlet et le potentiel de simple couche" (continuation and conclusion) by G. Bertrand, 298–307; "Sur la dérivation et l'intégration génèralisées" by P. Levy, 307–320.

BULLETIN OF THE AMERICAN MATHEMATICAL SOCIETY, volume 29, November, 1923: "An electro-magnetic theory of light-darts" by H. Bateman, 385–393; "Some left co-set and right co-set multipliers for any given finite group" by G. A. Miller, 394–398; "The second mean value theorem for summable functions" by M. B. Porter, 399–400; "Analogies between the U_n, V_n of Lucas and elliptic functions" by E. T. Bell, 401–406; "Singularities of curves of given order" by T. R. Hollcroft, 407–414; "The Hurwitz-Courant Funktionentheorie" [review of A. Hurwitz and R. Courant, *Vorlesungen über allgemeine Funktionentheorie und elliptische Funktionen* (Berlin, 1922)] by O. D. Kellogg, 415–417; Reviews: by A. R. Crathorne of C. V. L. Charlier, *Vorlesungen über die Grundzüge der mathematischen Statistik* (2nd ed., Lund), 418–419; by P. J. Daniell of M. Milankovitch, *Théorie Mathématique des Phénomènes Thermiques produits par la Radiation Solaire* (Paris, 1920), 419–420; by F. Cajori of G. Monge, *Géométrie Descriptive* (Paris, 1922), 420; by L. Ingold of L. Bieberbach, *Differentialrechnung* (Leipzig and Berlin, 1922), 421; by H. B. Phillips of M. Moller, *Kraftarten und Bewegungsformen* (Braunschweig, 1922), 421; by C. H. Forsyth of H. E. Soper, *Frequency Arrays* (Cambridge, 1922), 422; by E. P. Adams of *Atomes et Electrons* (Paris, 1923), 422–423; by H. B. Phillips of A. Gray, G. B. Mathews, and

T. M. MacRobert, *Bessel Functions* (London, 1922), 423; by L. C. Karpinski of H. Wieleitner, *Geschichte der Mathematik* (I, Berlin, 1922), 424; and by E. B. Cowley of A. Mac Leod, *Introduction à la Géométrie non-Euclidienne* (Paris, 1922), and of H. Liebmann, *Nichteuklidische Geometrie* (Berlin and Leipzig, 1923), 425; Notes, 425–428; New publications, 429–432.

MATHEMATICS TEACHER, volume 16, October, 1923: "Calculus in the high school" by J. M. Kinney, 321–331; "Time is money" by W. E. Breckenridge, 332–334; "Public honors for mathematical contributions" by G. A. Miller, 335–339; "A Dutch text-book of 1730" by L. G. Simons, 340–347; "Magic circles" by Vera Sanford, 348–349; "Systematic procedure in the solution of algebraic problems" by R. R. Goff, 350–355; "The price of wisdom" by G. W. Evans, 356–358; "Japanese problems" by Shige Hiuama, 359–365; "A study in fractions" by J. E. Worthington, 366–373; News and notes, 374–381; New books, 382–384,—November: "Mathematics club program" by A. H. Wheeler, 385–390; "Craig's edition of Euclid: its use and application of the principal propositions given" by Agnes G. Rowlands, 391–397; "The origin of our numerals" by C. P. Sherman, 398–401; "Live problem material in algebra" by D. S. Davis, 402–413; "Measuring achievement in first year algebra" by H. R. Douglass, 414–420; "Advantages of a general course in mathematics for the first two years in high school" by L. A. McCoy, 421–424; "On the precedence of numerical operations" by R. E. Moritz, 425–430; "The place of the Calculus in the training of the high school teacher" by B. Cosby, 431–439; Discussion, 440–442; News and notes, 443–446; New books, 447–448.

MATHEMATISCHE ZEITSCHRIFT, volume 18, nos. 1–2, September, 1923: "Beitrag zu den Grundlagen der kombinatorischen Analysis situs" by E. Bilz, 1–41; "Eine neue Behandlung der ersten Randwertaufgabe für $\Delta u = 0$" by O. Perron, 42–54; "Ein Seitenstück zur Theorie der linearen Transformationen einer komplexen Veränderlichen. I." by E. Study, 55–86; "Ueber die Randwerte einer analytischen Funktion" by F. Riesz, 87–95; "Herleitung des Gausschen Fehlergesetzes aus einer Funktionalgleichung" by G. Polya, 96–108; "Ueber dyadische Brüche" by A. Khintchine, 109–116; "Ueber eine Verallgemeinerung der Parsevalschen Formel" by F. Riesz, 117–124; "Ueber das Eulersche Summierungsverfahren" (II Mitteilung) by K. Knopp, 125–156; "Ueber eine Verallgemeinerung der Eulerschen Reihentransformation" by O. Perron, 157–172.

MESSENGER OF MATHEMATICS, volume 53, no. 1, May, 1923: "On certain puzzle-questions occurring in early arithmetical writings and the general partition problems with which they are connected" by J. W. L. Glaisher, 1–16.

PROCEEDINGS OF THE NATIONAL ACADEMY OF SCIENCE OF THE U. S. A., volume 9, November, 1923: "A generalization of Volterra's derivative" by H. L. Smith, 397–398.

QUARTERLY JOURNAL OF MATHEMATICS, volume 49, No. 4, March, 1923: "On Legendre's coefficients and associated functions with non-integral subscripts, and their connection with the elliptic integrals" by H. B. C. Darling, 289–303; "On certain Riccati differential equations with algebraic integrals" by A. Berry, 303–308; "Some notes on spheroidal wave-functions" by E. G. C. Poole, 309–321; "Singly infinite class number relations" by E. T. Bell, 322–337; "The genesis of Lamé's equation" by E. H. Neville, 338–352; "Chess tournaments and the like treated by the calculus of symmetric functions" by P. A. MacMahon, 353–384.

TRANSACTIONS OF THE AMERICAN MATHEMATICAL SOCIETY, volume 24, October, 1922 [published, November, 1923]: "On the location of the roots of certain types of polynomials" by J. L. Walsh, 163–180; "A note on the preceding paper" by D. R. Curtiss, 181–184; "Determination of all general homogeneous polynomials expressible as determinants whose elements are homogeneous polynomials" by H. S. Everett, 185–194; "General vector calculus" by J. B. Shaw, 195–244.

PROBLEMS AND SOLUTIONS.

EDITED BY B. F. FINKEL, OTTO DUNKEL, AND H. L. OLSON.

Send all communications about Problems and Solutions to B. F. FINKEL, Springfield, Mo.

PROBLEMS FOR SOLUTION.

[N. B. Problems containing results believed to be new, or extensions of old results, are especially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the state-

ments. In general, problems in well-known text-books, or results found in readily accessible sources, will not be proposed as problems for solution in the MONTHLY. In so far as possible however, the editors will be glad to assist the members of the Association with their difficulties in the solution of such problems.]

3055. Proposed by J. L. RILEY, Tarleton Station, Texas.

Given the non-intersecting circles

$$x^2 + y^2 + a_1x + b_1y + c_1 = 0,$$
$$x^2 + y^2 + a_2x + b_2y + c_2 = 0;$$

it is required to find the four common tangents.

3056. Proposed by W. H. RASCHE, Virginia Polytechnic Institute.

Given four points on the surface of a sphere determining a spherical quadrangle of sides a, b, c, d respectively. Let α be the angle between sides d and a, and β the angle between sides c and d. Give the relation between the angles α and β in terms of a, b, c, d. This is a fundamental problem in connection with spherical mechanisms and linkages in general.

3057. Proposed by A. A. BENNETT, University of Texas.

Euler in a letter to Stirling stated (without any hint as to the method of proof) that $\pi/4$ might be represented as an infinite product where the nth factor is the quotient of the nth odd prime when divided by the integral multiple of 4 nearest to it. Prove that

$$\frac{\pi}{4} = \frac{3\cdot5\cdot7\cdot11\cdot13\cdot17\cdot\cdots}{4\cdot4\cdot8\cdot12\cdot12\cdot16\cdot\cdots}.$$

3058. Proposed by LOUIS WEISNER, University of Rochester.

If n is any integer greater than 1, the number of integers less than n and prime to n of the form $c + xd$ is $\phi(n)/\phi(d)$, where d is a divisor of n and c is an integer less than d and prime to d.

3059. Proposed by DANIEL KRETH, Wellman, Iowa.

Given the perimeter and the radii of the inscribed and circumscribed circles, to construct the triangle and calculate the lengths of its sides.

3060. Proposed by B. H. BROWN, Dartmouth College.

A link-work of 3 bars, OA, AB, and BO' is fastened to a plane at O and O'; $OO' = AB = 2a$, $OA = BO' = \sqrt{2}a$. The locus of P, the mid-point of AB, is a circle when the bars form (with OO') a parallelogram; and a lemniscate when the bars form a contra-parallelogram.[1] In the latter case, show that the locus of Q, the intersection of OA and $O'B$ (produced) is an equilateral hyperbola. Generalize when $OA = BO' = b$. Consider the relation of the points P and Q from the standpoint of the Argand plane.

3061. Proposed by NORMAN ANNING, University of Michigan.

The cubic, $48y = 25x - x^3$ fits the sine curve, $y = \sin(\pi x/6)$ for $x = -3, -1, 0, 1, 3$. For what x $(-3 < x < 3)$ is the fit the poorest?

SOLUTIONS.

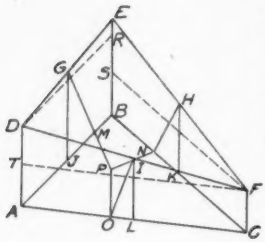
2982 [1922, 313]. Proposed by H. P. MANNING, Providence, R. I.

Solve the following problem without using analytical geometry: A triangular yard has a post at each corner. Given the lengths of the sides of the yard and of the posts, find the length and the position of the foot of a ladder that will just reach to the top of each post without changing the position.

SOLUTION BY B. F. FINKEL, Drury College.

Let ABC be the yard and AD, BE, and CF the posts. Connect the tops of the posts forming the triangle DEF. Let G, H, and I be the mid-points of the sides DE, EF, and FD, respectively. In the plane $ABED$, draw GM perpendicular to DE intersecting AB at M. In like manner,

[1] Link-Work for the Lemniscate, *American Journal of Mathematics*, 1878, vol. 1, p. 386.



determine the point N in the line BC. The point M is equidistant from the points D and E and the point N is equidistant from the points E and F. In the plane of the triangle ABC, erect a perpendicular, MP, to AB at M and a perpendicular, NP, to BC at N. Then all points in MP are equidistant from the points D and E and all points in NP are equidistant from the points E and F. Hence, P, the intersection of MP and NP, is equidistant from the points D, E, and F. Hence, P is the position of the foot of the ladder.

To compute the length of the ladder, let $AB = c$, $AC = b$, $BC = a$, $AD = d$, $BE = e$, and $CF = f$. Draw GJ perpendcular to AB, HK perpendicular to BC, IL perpendicular to AC, DR perpendicular to BE, FS perpendicular to BE, and FT perpendicular to AD.

Then $GJ = (d + e)/2$, $HK = (e + f)/2$, and $IL = (f + d)/2$. From the similar triangles, GJM and ERD, we find $MJ = (e^2 - d^2)/2c$ and $MB = (c^2 + d^2 - e^2)/2c$. Similarly, we find $BN = (a^2 + f^2 - e^2)/2a$.

In the quadrilateral $BMPN$, since the angles at M and N are right angles, the quadrilateral is inscribable in a circle, from which it follows that $BP = MN/\sin B$, but $MN^2 = BM^2 + BN^2 - 2BM \cdot BN \cos B$. Hence,

$$BP^2 = (BM^2 + BN^2 - 2BM \cdot BN \cos B)/\sin^2 B$$

and the length of the ladder, l, is

$$l = \sqrt{BE^2 + BP^2}$$

$$= \sqrt{e^2 + \left[\left(\frac{c^2 + d^2 - e^2}{2c}\right)^2 + \left(\frac{a^2 + f^2 - e^2}{2a}\right)^2 - 2\,\frac{c^2 + d^2 - e^2}{2c} \cdot \frac{a^2 + f^2 - e^2}{2a} \cos B\right] / \sin^2 B}$$

$$= \sqrt{e^2 + \frac{a^2(c^2 + d^2 - e^2)^2 + c^2(a^2 + f^2 - e^2)^2 - (c^2 + d^2 - e^2)(a^2 + f^2 - e^2)(a^2 + c^2 - b^2)}{16s(s - a)(s - b)(s - c)}}$$

where $2s = a + b + c$.

If $d = e = f = 0$, $l = \dfrac{abc}{4A}$, where A is the area of the triangular yard.

Also solved by H. N. Carleton, A. Pelletier, J. B. Reynolds, and F. E. Wood.

2993 [1922, 420]. Proposed by H. C. Bradley, Massachusetts Institute of Technology.

Let ABC be any triangle, and O the center of its circum-circle. Bisect the arcs AB, BC, and CA at F, D, and E. With F, D, and E as centers draw arcs passing in each instance through the adjacent corners of the triangle. Prove that these arcs intersect at the in-center of the triangle ABC.

I. Solution by Theodore Bennett, University of Illinois.

We shall first establish the following lemma: Let $A_1A_2A_3$ be the vertices of any triangle, and let $C_1C_2C_3C_4$ be the centers of the four circles which touch the three sides of $A_1A_2A_3$. Let P_{ij} be the mid-point of the segment C_iC_j. Then the points P_{ij} lie on the circum-circle of $A_1A_2A_3$.

It is clear that any one of the points C_i is the ortho-center of the triangle (T_i) formed by the other three, and the feet of the altitudes of the triangle T_i are the points A_i. Hence, the nine point circle of T_i is the circle passing through $A_1A_2A_3$. From the well-known properties of the nine point circle, it follows that this circum-circle of $A_1A_2A_3$ bisects each segment C_iC_j, and thus passes through the six points P_{ij}.

Now consider the triangle ABC, with in-center I, and circum-circle O. Construct the circle K, passing through ABI. The lines AI and BI are two of the bisectors of the angles at A and B; let the other bisectors of these angles be AJ and BJ. Then it is obvious that the center of K is the mid-point of IJ. But I and J are two of the points C_i in our lemma, and therefore the center of K lies on the circle O, and is obviously F, the mid-point of the arc AB. Similarly the circles described on BC and CA pass through I, and the theorem is proved.

It should be noted that there are two possible positions for each of the points F, E, and D, giving rise to six circles, which pass by threes through the four points C_i, each of these six circles containing two of the points C_i.

II. Solution by H. Halperin, College Station, Tex.

The lines AD, BE, and CF are the bisectors of the angles A, B, and C of the triangle ABC and hence intersect at O', the in-center of the triangle. Consider triangle AFO'. The angle $FO'A$ is measured by $\frac{1}{2}$(arc AF + arc DC) and the angle $O'AF$ is measured by $\frac{1}{2}$(arc FB + arc BD). But arc AF = arc FB and arc DC = arc BD, hence $\angle FO'A = \angle O'AF$ and hence $FO' = FA = FB$ and O' lies on the arc AB described with F as center. Similarly it is proved that it lies on the arcs BC and CA, described with D and E as centers, respectively.

Also solved by Michael Goldberg, William Hoover, A. Pelletier, A. V. Richardson, and W. W. Weber.

In his article "On the I-Centers of a Triangle" (*1918*, 241–246), Professor Altshiller-Court gave a simple proof of the theorem contained in this problem and references to previous proofs.

2994 [1922, 420]. Proposed by R. M. Mathews, Wesleyan University.

Can the following construction be made without the use of a regulus? Construct a line which meets four given skew lines.

Solution by Elijah Swift, University of Vermont.

Yes, the line required can be found by the usual Euclidean constructions,—passing a plane through a point and a line, and in the plane using a straightedge and one fixed circle.

Let the lines be α. β, γ, and δ. Take any point, P, on α and pass a plane through P and β cutting δ in B and a second plane through P and γ cutting δ in C. Then the line PB cuts α, β, and δ and the line PC cuts α, γ, and δ. If we can find a position of P for which B and C coincide, the line $PB = PC$ corresponding will cut all four given lines.

If we let the point, P, take different positions, P_1, P_2, etc., we obtain two ranges of points on δ, B_1, B_2, etc., and C_1, C_2, etc. These ranges are projective, for they are the intersections of δ with pencils of planes through β and γ, respectively, and corresponding planes of the two pencils cut α in the same point, P. The problem now reduces to finding the self-corresponding points of two projective ranges on the same line determined by three corresponding pairs (B_1, C_1), (B_2, C_2), (B_3, C_3), and this may be solved by the well-known construction with a fixed circle. Since there may be two, one or no such points, there may be two, one or no lines cutting all four given skew lines.

2995 [1922, 420]. Proposed by S. A. Corey, Des Moines, Iowa.

Give a geometric proof of each of the identities

(*a*) $\cos(a + 2mx) = \cos a - 2\sin x[\sin(a + x) + \sin(a + 3x) + \cdots + \sin(a + (2m - 1)x)]$

(*b*) $\sin(a + 2mx) = \sin a + 2\sin x[\cos(a + x) + \cos(a + 3x) + \cdots + \cos(a + (2m - 1)x)]$

where m is a positive integer.

Solution by A. V. Richardson, Bishop's College.

Let $P'OP$ be the horizontal diameter of a circle in the positive direction with center O and radius $OP = 1$ Take on the circumference the points A, A_2, A_4, $\cdots$, A_{2m} so that $\angle POA = a$, $\angle AOA_2 = \angle A_2OA_4 = \cdots = \angle A_{2m-2}OA_{2m} = 2x$; and denote by Q_{2i} the projection of A_{2i} on the diameter $P'P$. Now AA_2 makes with OP (the positive direction) the angle $90° + a + x$, as is readily seen by dropping the perpendicular OE_1 upon AA_2. Hence

$$QQ_2 = AA_2\cos(90° + a + x) = -2EA_2\sin(a + x) = -2\sin x\sin(a + x)$$

Similarly $Q_{2i-2}Q_{2i} = -2\sin x\sin[a + (2i - 1)x]$. Now the identity $OQ_{2m} = OQ + QQ_2 + Q_2Q_4 + \cdots + Q_{2m-2}Q_{2m}$ gives at once $\cos(a + 2mx) = \cos a - 2\sin x\sin(a + x) - 2\sin x\sin[a + 3x] - \cdots - 2\sin x\sin[a + (2m - 1)x]$. By projecting on the vertical diameter we obtain the expansion for $\sin(a + 2m\,x)$.

2996 [1922, 420]. Proposed by E. J. OGLESBY, Flushing, N. Y.

Given $u_1 = .2500$, $u_2 = .4113$, $u_3 = .4785$, $u_4 = .4965$, find x when $u_x = .4311$.

SOLUTION BY S. A. COREY, Des Moines, Iowa.

This problem admits of more than one solution. If we assume that u_x is an analytic function of x whose fourth and higher derivatives are zero, a solution may be obtained as follows.

Let $y = x - 1$. Then we may write $u_y = .2500 + ay + by^2 + cy^3$, where the values of a, b and c may be found from the equations obtained by giving u the values stated. Substituting these values of a, b and c ($.2233\frac{1}{3}$, $-.0695$, $.0074\frac{2}{3}$) in the above equation, we obtain a cubic whose only real root is $y = 1.203$. Whence $x = 2.203$.

Also solved by H. HALPERIN, A. PELLETIER, J. L. RILEY, ELIJAH SWIFT, and the PROPOSER.

2997 [1922, 420]. Proposed by M. ZAMETKIN, Jamaica, New York.

Given $a = \sin 5°$, $b = \sin 49°$, and $c = \sin 87°$, prove that

$$\sin 73° = \frac{a^2 - b^2 + ac}{4a(a^2 - b^2 + ac) - (a - b + c)}.$$

SOLUTION BY MAURICE BAUDIN, Miami University.

It is found by successive reductions that

$$a^2 - b^2 + ac = -\sin 39° \sin 49°;$$

and then that

$$a - b + c = \frac{a^2 - b^2 + ac + bc}{a + b} = \frac{\sin 49° \sin 24°}{\sin 68°} = 4 \sin 49° \sin 8° \sin 52°.$$

Hence we have

$$\frac{a^2 - b^2 + ac}{4a(a^2 - b^2 + ac) - (a - b + c)} = \frac{\sin 39°}{4(\sin 5° \sin 39° + \sin 8° \sin 52°)} = \sin 73°,$$

since $\sin 39° = 4 \sin 13° \sin 47° \sin 73°$, and $\sin 5° \sin 39° + \sin 8° \sin 52° = \sin 13° \sin 47°$.

Also solved by H. HALPERIN, A. PELLETIER, and A. V. RICHARDSON.

NOTES AND NEWS.

It is hoped that readers of the MONTHLY will coöperate in contributing to the general interest of this department by sending items to R. W. BURGESS, Brown University, Providence, R. I.

Professor L. A. HOWLAND, head of the department of mathematics at Wesleyan University, has been made acting president of that institution.

Assistant Professor MARY CURTIS GRAUSTEIN, of Wellesley College, has returned to the college after two years' leave of absence.

At the University of Vermont, Mr. H. G. MILLINGTON has been promoted to an assistant professorship of mathematics, and has been transferred from the College of Arts and Sciences to the College of Engineering. Mr. G. H. NICHOLSON, of Harvard University, has been appointed instructor of mathematics in the Arts College.

At Colgate University, Mr. H. A. DOBELL has been granted leave of absence for study at the University of Pennsylvania, and Mr. J. C. POLLEY, of Yale University, and Mr. G. O. HOHL, of Lebanon Valley College, have been appointed instructors.

At the Carnegie Institute of Technology, Professor O. T. GECKELER has been made head of the department of mathematics; Assistant Professor H. S. LIGHTCAP has been promoted to an associate professorship, and Mr. R. P. JOHNSON, Mr. E. A. WHITMAN, and Mr. E. A. STARR have been promoted to assistant professorships. Mr. J. H. SIMESTER of the University of Toronto has been appointed instructor.

At Swarthmore College, Mr. L. J. COMRIE has been promoted from the position of research assistant in the Observatory to an assistant professorship of mathematics and astronomy. Professor J. A. MILLER lectured October 25 on the Solar Eclipse of 1923 before the Franklin Institute.

At Lafayette College, Associate Professor W. M. SMITH has been granted a six months' leave of absence for study at Oxford and Cambridge. Mr. E. H. WELLS, formerly of Wooster University, has been appointed instructor.

At St. John's College, Annapolis, Maryland, Professor B. H. WADDELL and Assistant Professor T. L. GLADDEN have resigned, and Mr. G. A. BINGLEY, instructor of mathematics at the United States Naval Academy, has been appointed assistant professor.

Mr. F. W. WINTERS, of Harvard University, has been appointed assistant professor of mathematics at Miami University.

Mr. R. H. MACCULLOUGH has been appointed head of the department of mathematics at Defiance College (Ohio).

Miss NORMA STELFORD, of the State Normal School at Superior, Wisconsin, has accepted a position on the staff of the Northern Illinois State Teachers College, DeKalb, Illinois.

Professor H. P. KEAN, of Illinois Wesleyan University, has been appointed assistant professor of mathematics at Wittenberg College, Springfield, Ohio.

Dr. H. P. PETTIT, of the University of Illinois, has been appointed professor of mathematics and head of the department at Illinois Wesleyan University.

At Purdue University, Dr. W. E. EDINGTON has been promoted to an assistant professorship, and Mr. C. E. BARR has been appointed instructor of mathematics.

Dr. MARIE WHELAN, of Johns Hopkins University, has been appointed professor of mathematics at Olivet College (Michigan).

At the Michigan State Normal College, Ypsilanti, Dr. THEODORE LINDQUIST, formerly head of the department of mathematics, State Teachers' College, Emporia, Kansas, has been appointed professor of mathematics, and Associate Professor J. F. BARNHILL has been promoted to a full professorship.

At Beloit College, Mr. R. E. HUFFER, formerly instructor at Michigan Agricultural College, has been appointed assistant professor; Mr. WILLIAM OLIVER has been appointed instructor in applied mathematics, and Miss IRENE ELDRIDGE has been given a temporary appointment as instructor.

At Kentucky Wesleyan College, Mr. W. B. HUGHES has been appointed to succeed Professor R. G. DEMAREE as head of the department.

At Erskine College, Due West, South Carolina, Mr. H. A. WISE of the University of South Carolina has been appointed to succeed Professor W. C. HALLIDAY as professor and head of the department.

At Lander College, Greenwood, South Carolina, Professor W. W. WEBER, formerly head of the department of mathematics at Southern College, Lakeland, Florida, has been appointed to succeed Professor J. D. SALLEY, and Miss LAURA D. SARGENT has been appointed associate professor of mathematics to succeed Mr. ERROL MARTIN.

Professor C. S. COX, formerly head of the department of mathematics and physics at Birmingham Southern College, has been appointed to the same position at Southern College, Lakeland, Florida.

At the University of Georgia, Associate Professor D. F. BARROW has been promoted to a full professorship; Instructors A. H. STEVENS and J. P. HILL have resigned, and Mr. FOREST CUMMING and Mr. E. M. EVERETT, both of the University of Georgia, have been appointed instructors.

At the Georgia School of Technology, instructors E. R. C. MILES, R. L. DRISCOLL, P. L. ARMSTRONG, E. B. WILSON, and G. T. TRAWICK have resigned, and the following have been appointed instructors: J. G. EVANS, W. W. PURKS, J. G. GRIFFIN, T. P. BRANCH, J. W. TAYLOR.

At Iowa State College, Assistant Professor E. C. KIEFER has resigned to become head of the department of mathematics at James Millikin University. Mr. B. A. ROGERS, formerly professor of physics at the Connecticut College of Agriculture, has been appointed instructor.

At Washington University, St. Louis, Assistant Professor THEODORE DOLL has resigned to accept a position with the American Bridge Company, and will be located at Gary, Indiana. Instructor M. E. MEYERSON has resigned to take up work in commercial chemistry. Dr. JESSICA M. YOUNG has been promoted to an assistant professorship of mathematics and astronomy. Mr. EDMOND SIROKY, of the Terrell Croft Engineering Company, has been appointed assistant professor of applied mathematics. Mr. H. A. HOOVER, of Iowa State College, has been appointed instructor of mathematics.

Associate Professor S. LEFSCHETZ, of the University of Kansas, has been promoted to a full professorship of mathematics.

Mr. A. S. HATHAWAY has been appointed professor of mathematics at Friends University, Wichita, Kansas.

Associate Professor W. T. STRATTON of Kansas State Agricultural College has been promoted to a full professorship of mathematics.

At Creighton University, Omaha, Nebraska, Professor PERK has gone to St. Louis University for higher studies. Mr. PATRICK REGAN, of St. Louis University, and Mr. L. E. ROMBAUT, of Notre Dame University, have been appointed assistant professors of mathematics. Mr. ALPHONSE SCHMITT of St. Louis University has been serving as professor since September, 1922.

Professor D. A. LEHMAN, of Goshen College, has been appointed professor of mathematics at Bluffton College.

At the University of North Dakota, Instructor J. D. LEITH has been granted leave of absence for the current academic year and is doing graduate work at Columbia University. Mr. S. F. BIBB of the University of Chicago and Mr. J. F. GATES of the University of North Dakota have been appointed instructors.

At the Colorado Agricultural College, Instructor C. W. YOUNG has been promoted to an assistant professorship, and Mr. A. G. CLARK, instructor of mathematics at the University of Wyoming, has been appointed to an assistant professorship.

At the University of Oklahoma, Professor S. W. REAVES, head of the department of mathematics, has been appointed acting dean of the College of Arts and Sciences. Associate Professor J. O. HASSLER has been promoted to a full professorship; Assistant Professor N. A. COURT has been promoted to an associate professorship. Instructors ELLA MANSFIELD and H. G. LIEBER have resigned, the latter to continue his graduate studies at Columbia University.

At Oklahoma City College, Mr. C. M. ALLEN, formerly head of the department of mathematics, has been transferred to the Education Department. He is succeeded as head of the department of mathematics by Mr. G. E. MEADER, formerly assistant professor of Education at Oklahoma University.

At Oklahoma Agricultural and Mechanical College, Mr. C. B. GRUMBINE has been appointed professor of mathematics and Mr. W. C. PAYNE assistant professor.

Associate Professor J. W. CALHOUN, of the University of Texas, has been promoted to a full professorship of applied mathematics.

Mr. W. M. WHYBURN, of the University of Texas, has been appointed professor of mathematics at South Park College, Beaumont, Texas.

At the Agricultural and Mechanical College of Texas, Assistant Professor W. L. HUGHES and Instructor F. W. FRARY have resigned; Mr. F. W. SPARKS has been appointed assistant professor, and Dr. W. P. UDINSKI and Mr. W. D. BATEN have been appointed instructors.

The following appointments to instructorships of mathematics at American colleges and universities are announced: At Case School of Applied Science, Mr. V. I. BENANDER, of Harvard University; at the University of Florida, Mr. H. W. CHANDLER of the University of Minnesota; at Hamline University, Mr. E. I. MICKELSON of the University of Minnesota; at Illinois College, in mathematics and physics, Mr. G. W. SCHNEIDER, succeeding Mr. W. G. GUILD, now continuing his graduate study at the University of Illinois; at John Tarleton Agricultural College, Miss MARY MARRS; at Keuka College, Mr. C. I. KELCHNER; at Meredith College, Miss MARGARET WYATT succeeding Mrs. E. M. HIGHSMITH; at the University of Rochester, Dr. LOUIS WEISNER of Columbia University; at Rockford College, Miss MARTHA P. MCGAVOCK, of Wellesley College; at the University of Tennessee, Mr. AUGUSTUS LISK, of the University of Kentucky; at Tulane University, Mr. C. G. LATIMER and Mr. T. F. COPE; at Union College, Mr. H. N. ROWE, succeeding Mr. R. D. BENNETT, who resigned for further study at the University of Chicago; at Virginia Polytechnic Institute, Mr. T. A. HATCHER; at Washington State College, Mr. O. M. AKEY, of Ohio State and Stanford Universities, succeeding Mr. R. B. KENNEDY; at West Virginia University, Mr. CLAIRE HARKINS.

LADY SHAW, wife of SIR NAPIER SHAW, died September 22, 1923. She was at one time lecturer in mathematics at Newnham College, Cambridge.

Professor M. A. BAILEY, head of the department of mathematics at the New York Training School for Teachers since 1899, died on November 25th at the age of sixty-seven. After headmasterships at the high schools in Winsted, Conn., and Keene, N. H., he went to the State Normal School at Emporia, Kansas, as organizer and supervisor of the mathematics department. For the last twenty-four years he prepared the curriculum in mathematics for the New York schools.

The degree of doctor of laws was conferred on Professor R. C. ARCHIBALD on the occasion of the inauguration of President G. J. TRUEMAN at Mount Allison University, October 18.

Mr. HUGH PORTER, who has been principal of the senior high school and dean of the junior college at Wichita Falls, Texas, has returned to the North Texas State Normal College as professor of mathematics.

Professor J. A. WHITTED has resigned his position at Hedding College and has become professor of mathematics at Ohio Northern University.

Associate Professor W. P. RUSSELL of Pomona College is spending a year's leave of absence in study at Harvard.

Mr. C. B. WALSH, headmaster of Woodmere Academy, Woodmere, N. Y., died on July 13, 1923, at the age of thirty-nine years. Mr. Walsh was a charter member of this Association and was, for a time, president of the Association of Teachers of Mathematics of the Middle States and Maryland.

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